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C H A P T E R 9

Exponential and Logarithmic Functions

This chapter discusses two general classes of functions, exponential functions and logarithmic functions. Each is the inverse of the other, and both have many applications in banking (compound interest), biology (population growth), physics (chain reactions), electronics (charging a resistance-capacitance network), psychology (learning curves), and computer science (the height of a balanced binary tree). Both functions also have many other applications in these and other areas.

9–1 *Exponential functions and their properties*

> If m represents the visual magnitude of a star, then the ratio R of the brightness of the star to a star of the first magnitude is approximately $R(m) = 2.5^{1-m}$. Graph this function.

This function, called an exponential function, is the topic of this section.

Real number exponents

We previously defined integral and rational exponents. For example,

$2^3 = 2 \cdot 2 \cdot 2 = 8$	Positive integer exponent
$2^0 = 1$	Zero exponent
$2^{-3} = \frac{1}{8}$	Negative integer exponent
$2^{\frac{1}{7}} = \sqrt[7]{2} \approx 1.104$	Rational exponent

We have not defined the meaning of exponents that are irrational, such as π or $\sqrt{2}$. For example, what would 2^π mean? Since $\pi \approx 3\frac{1}{7}$, we know $2^\pi \approx 2^{3\frac{1}{7}}$, which is $2^3 \cdot 2^{\frac{1}{7}} = 8\sqrt[7]{2}$.

This shows how we can approximate values for a base with an irrational exponent by considering a rational number with a value close to the irrational value. In more advanced mathematics it is possible to define exponents with

irrational values in a manner similar to this. For the purposes of this text, we assume that this definition has been made.

It can be proved that the properties of exponents that are true for integer and rational exponents hold for any real exponent. Some of these properties are summarized here.

Properties of exponents

If $x, y, z \in R$, then

[1]	$x^y \cdot x^z = x^{y+z}$	[2]	$\frac{x^y}{x^z} = x^{y-z}, x \neq 0$	[3]	$(xy)^z = x^z y^z$
[4]	$\left(\frac{x}{y}\right)^z = \frac{x^z}{y^z}$	[5]	$(x^y)^z = x^{yz}$	[6]	$x^{-y} = \frac{1}{x^y}$

Example 9–1 A illustrates the properties of exponents.

■ ***Example 9–1 A***

Use the properties of exponents to simplify each expression.

1. $3^\pi \cdot 3^2$
$= 3^{\pi+2}$

2. $(5^{\sqrt{2}})^{\sqrt{8}}$
$= 5^{(\sqrt{2}\cdot\sqrt{8})}$
$= 5^{\sqrt{16}} = 5^4$
$= 625$

3. $\frac{\pi^{\sqrt{8}}}{\pi^{\sqrt{2}}}$
$= \pi^{(\sqrt{8}-\sqrt{2})}$
$= \pi^{(2\sqrt{2}-\sqrt{2})}$
$= \pi^{\sqrt{2}}$ ■

Exponential function—definition

With the knowledge that any real exponent has meaning, we can now define a class of functions in which the domain element is the exponent of a fixed base.

Exponential function

An exponential function is a function of the form

$$f(x) = b^x, \quad b > 0 \quad \text{and} \quad b \neq 1$$

The constant value b is called the **base** of the function. The variable x can represent any real number, and therefore the domain of an exponential function is the set of real numbers.

The function $f(x) = 5^x$ is an example of an exponential function. The value of f for various domain elements is computed.

$$f(2) = 5^2 = 25$$
$$f(0) = 5^0 = 1$$
$$f(-3) = 5^{-3} = \frac{1}{125}$$

Graphs of exponential functions

As further examples of the graphs of exponential functions consider the graph of $f(x) = 2^x$, the exponential function with base 2, and the function $f(x) = 3^x$, the exponential function with base 3. Some of the (x,y) ordered pairs for these functions is shown in table 9–1. These values are plotted and connected by smooth curves in figure 9–1. Information for plotting the graphs on the TI-81 graphing calculator is also shown. Observe that both of these functions have the same y-intercept, (0,1). This is because any base, raised to the zero power, is one. Also, $3^x > 2^x$ for $x > 0$, and $3^x < 2^x$ for $x < 0$.

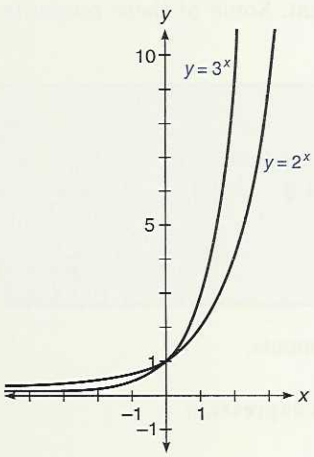


Figure 9–1

x	2^x	3^x
-3	$\frac{1}{8}$	$\frac{1}{27}$
-2	$\frac{1}{4}$	$\frac{1}{9}$
-1	$\frac{1}{2}$	$\frac{1}{3}$
0	1	1
1	2	3
2	4	9
3	8	27

Table 9–1

$\boxed{Y=}$ 2 $\boxed{\wedge}$ $\boxed{X|T}$ $\boxed{ENTER}$ 3 $\boxed{\wedge}$ $\boxed{X|T}$

$\boxed{RANGE\ -4,4,-1,10}$

The x-axis is a **horizontal asymptote** for both curves. Neither function has an x-intercept since $0 = b^x$ has no solution. Also, as x gets greater, the $y = f(x)$ values for both curves keep getting greater also so the function is increasing (section 3–5).

The graphs in figure 9–1 illustrate the behavior of exponential functions for $b > 1$. When $0 < b < 1$ we get graphs similar to these, but that are decreasing. Table 9–2 and figure 9–2 illustrate the curves for the functions $f(x) = (\frac{1}{2})^x$ and for $f(x) = (\frac{1}{3})^x$.

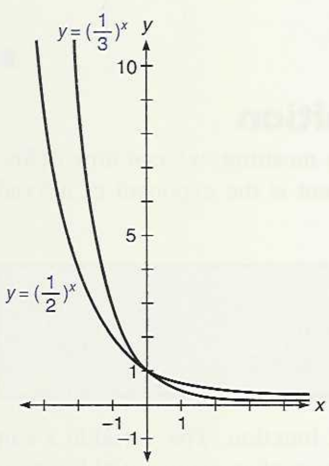


Figure 9–2

x	$(\frac{1}{2})^x$	$(\frac{1}{3})^x$
-3	8	27
-2	4	9
-1	2	3
0	1	1
1	$\frac{1}{2}$	$\frac{1}{3}$
2	$\frac{1}{4}$	$\frac{1}{9}$
3	$\frac{1}{8}$	$\frac{1}{27}$

Table 9–2

$\boxed{Y=}$ .5 $\boxed{\wedge}$ $\boxed{X|T}$ $\boxed{ENTER}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\wedge}$ $\boxed{X|T}$

$\boxed{RANGE\ -4,4,-1,10}$

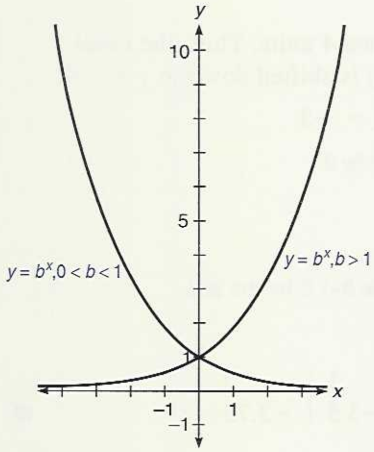


Figure 9–3

Figure 9–3 illustrates the general graphs of $f(x) = b^x$ for $b > 1$ and for $0 < b < 1$. When b is greater than 1 the graph is increasing, and when b is less than 1 the graph is decreasing. From the graphs of figure 9–3 we can see that *an exponential function is one to one.* We can see this because the graphs pass the horizontal line test (section 3–5). We can also observe from the graph that the domain of the exponential function is all the real numbers, and its range is all $y > 0$. This is summarized as follows.

Features of exponential function and its graph

An exponential function is a function of the form

$$f(x) = b^x, \quad b > 0 \quad \text{and} \quad b \neq 1, \text{ with}$$
$$\text{Domain: } \{x \mid x \in R\}$$
$$\text{Range: } \{y \mid y \in R, y > 0\}$$

- Exponential functions are one-to-one functions.
- The y-intercept is (0,1) (assuming no vertical or horizontal translations).
- The x-axis is a horizontal asymptote.
- If $b > 1$, the function is increasing.
- If $b < 1$, the function is decreasing.

To graph an exponential function using algebraic techniques we use the cases illustrated in figure 9–3 along with the observations we made earlier: translations (section 3–4) and point plotting. Of course the graphing calculator can be used as well. The appropriate information for the TI-81 is shown. This is all illustrated in example 9–1 B.

■ Example 9–1 B

Graph the function. State whether the function is increasing or decreasing. Note the value of any intercepts.

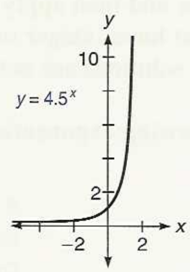
1. $f(x) = 4.5^x$

Since $4.5 > 1$ the function is an increasing function.

y-intercept: $\quad f(0) = 4.5^0 = 1$

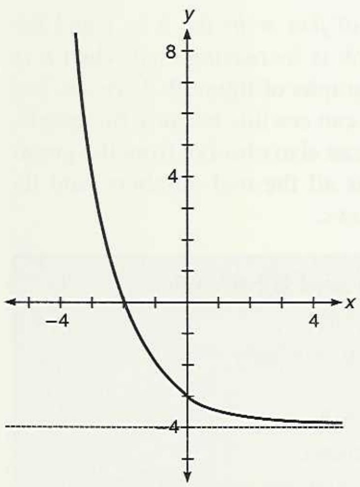
Additional points:

x	-1	1	1.5
$y = 4.5^x$	0.2	4.5	9.5



$\boxed{\text{Y=}}$ 4.5 $\boxed{\wedge}$ $\boxed{\text{X|T}}$

$\boxed{\text{RANGE } -2, 2, -1, 10}$



Y= 2 ∧ (−) X|T
− 4
RANGE −4,4,−5,8

2. $f(x) = 2^{-x} - 4$

This is the graph of $2^{-x} = (\frac{1}{2})^x$ shifted down 4 units. Thus, the usual horizontal asymptote of $y = 0$ (the x-axis) is shifted down to $y = -4$.

y-intercept: $f(0) = 2^0 - 4 = 1 - 4 = -3$

x-intercept:

$$
\begin{aligned}
0 &= 2^{-x} - 4 && \text{Replace } f(x) \text{ by } 0\\
4 &= 2^{-x}\\
2^2 &= 2^{-x}\\
2 &= -x\\
-2 &= x && \text{See example 9–1 C for this step}
\end{aligned}
$$

This is a decreasing function.
Additional values:

x	-4	-3	-1	1	2
$y = 2^{-x} - 4$	12	4	-2	-3.5	-3.75

Solving exponential equations with the one-to-one property

Recall from section 3–3 that a function is one to one if all of the second components of the ordered pairs are different. The fact that exponential functions are one to one implies the following about the situation where $b^m = b^n$. Suppose two ordered pairs (m,b^m) and (n,b^n) are in the exponential function $f(x) = b^x$, and $b^m = b^n$. Since the function is one to one, all second components are different. Therefore, if $b^m = b^n$ in the ordered pairs (m,b^m) and (n,b^n) (two second components are the same), these ordered pairs must actually be the same ordered pair. Thus, m and n must be the same, so we conclude that $m = n$. This fact implies the following statement.

One-to-one property for exponential functions

If $b^x = b^y$, then $x = y$.

Example 9–1 C illustrates using the one-to-one property to solve equations in which each member can be expressed as an exponential expression. Observe that *to solve these exponential equations we put both sides of the equation in terms of the same base* and then apply the one-to-one property. For now, we examine equations that have integer or rational solutions. Section 9–5 deals with equations whose solutions are not necessarily integer or rational.

■ Example 9–1 C

Solve for x in the following exponential equations.

1. $9^x = \frac{1}{27}$

$$
\begin{aligned}
(3^2)^x &= 3^{-3} && 9 = 3^2; \tfrac{1}{27} = \tfrac{1}{3^3} = 3^{-3}\\
3^{2x} &= 3^{-3} && (x^y)^z = x^{yz}\\
2x &= -3 && \text{One-to-one property}\\
x &= -\tfrac{3}{2} && \text{Divide both members by 2}
\end{aligned}
$$

2. $4^x = \sqrt{32}$

$4^x = \sqrt{2^5}$

$(2^2)^x = 2^{5/2}$ Rewrite both members as powers of 2

$2^{2x} = 2^{5/2}$ $(x^y)^z = x^{yz}$

$2x = \frac{5}{2}$ One-to-one property

$x = \frac{5}{4}$ ■

Mastery points

Can you

- State the features of an exponential function and its graph?
- Solve certain simple exponential equations?
- Use the properties of exponents to simplify expressions involving exponents?
- Graph certain exponential functions?

Exercise 9–1

1. Define the general equation of the exponential function with base b. Make sure you state the restrictions on b.

2. Describe the behavior of the function $f(x) = b^x$ with respect to the words "increasing" and "decreasing" and the value of b.

Use the properties of exponents to perform the indicated operations.

3. $2^\pi \cdot 2$ **4.** $5^{\sqrt{5}} \cdot 5^{\sqrt{20}}$ **5.** $7^\pi \cdot 7^{3\pi}$ **6.** $3^\pi \cdot 9^\pi$

$\boxed{7.}$ $\dfrac{4^{\sqrt{12}}}{4^{\sqrt{3}}}$ **8.** $\dfrac{2^{5\pi}}{2^{2\pi}}$ **9.** $(3^{\sqrt{2}})^{\sqrt{2}}$ **10.** $(3^{\sqrt{2\pi}})^{\sqrt{8\pi}}$

Solve the following exponential equations for x.

11. $3^x = 27$ **12.** $2^x = 512$ **13.** $3^x = \sqrt{27}$ **14.** $4^x = 2^5$

15. $9^3 = 3^x$ **16.** $10^{2x} = 1{,}000$ **17.** $2^x = \frac{1}{8}$ **18.** $4^x = \frac{1}{64}$

19. $4^x = \frac{1}{8}$ **20.** $8^x = \sqrt{128}$ **21.** $3^x = \sqrt[3]{243}$ **22.** $4^{x/2} = 8$

$\boxed{23.}$ $(\sqrt{2})^x = 16$ **24.** $2^{-x} = 0.25$

Graph each function. State whether the function is increasing or decreasing. Label any y-intercepts.

25. $f(x) = 5^x$ **26.** $f(x) = 6^x$ **27.** $f(x) = 4^{-x}$ **28.** $f(x) = 8^{-x}$

29. $f(x) = 0.3^x$ **30.** $f(x) = 0.9^x$ **31.** $f(x) = 0.7^{-x}$ **32.** $f(x) = 0.35^{-x}$

33. $f(x) = 4^{x+1}$ **34.** $f(x) = 3^{x-1}$ **35.** $f(x) = 3^{1-x}$ **36.** $f(x) = 2^{-x+1}$

$\boxed{37.}$ $f(x) = 4^{-x+2} + 1$ **38.** $f(x) = 2^{x/2} + 1$ **39.** $f(x) = 2^{-x}$ **40.** $f(x) = 3 - 3^x$

41. $f(x) = 2^{-x} + 2$ **42.** $f(x) = 2^x - 2$

Solve the following problems.

43. If a bank account paid 5% interest, compounded continuously, on the balance above \$10, then, if the initial amount deposited were \$13, the balance after time x (in years) would be closely described by the function $f(x) = 3(1.05)^x + 10$. Graph this function for $0 \le x \le 15$.

44. If x represents the strength of a sound in bels (one bel equals 10 decibels) then the factor F that represents the ratio of the given noise to a reference noise level is $F(x) = 10^x$. Graph this function.

45. If m represents the visual magnitude of a star, then the ratio R of the brightness of the star to a star of the first magnitude is approximately $R(m) = 2.5^{1-m}$. Graph this function.

46. A certain automobile seems to depreciate about 15% every year. Its value V after n years is given by the function $V = P(0.85^n)$, where P is the purchase price. Assume that P is 1 and graph the resulting function.

47. Assume the automobile of the previous problem cost \$12,000 when new. Approximate its value after 4 years.

48. The function $S = S_0 2.8^{-d}$ could describe the strength of a signal in a telephone cable at a distance d, measured in a suitable unit, from the source, where S_0 is the initial signal strength. Graph this function assuming that $S_0 = 1$.

49. In computer science, a binary tree has many uses. The number of pieces of data that can be stored in a binary tree of height h is $f(h) = 2^{h+1}$. Graph this function.

50. The value of 2^{10} is 1,024; this is close to 1,000, which is often indicated by the prefix "kilo." The amount of memory on a computer is often described in terms of kilobytes (a byte being one basic unit of memory). Thus, a 2 KB memory means a 2 kilobyte memory, or $2 \cdot 1{,}024 = 2{,}048$ bytes of memory. Find the exact number of bytes in the following values. (Note that 1,000 KB is often called 1 MB, for 1 megabyte.)
a. 16 KB **b.** 32 KB **c.** 512 KB **d.** 1,000 KB

51. As noted in problem 50, $2^{10} \approx 10^3$. This provides a way to estimate large powers of 2 in terms of powers of 10. For example, $2^{34} = 2^4 \cdot 2^{30} = 16(2^{10})^3 \approx 16(10^3)^3 = 16 \cdot 10^9$ (16 billion). Thus, 2^{34} is approximately 16 billion. Estimate the following values in terms of powers of 10.
a. 2^{13} **b.** 2^{21} **c.** 2^{43}

52. Exponential functions change rapidly for small changes in the value of the argument. (In $f(x) = 2^x$, x is the "argument.") We can illustrate this by a story. A person once saved the life of a certain very rich monarch. The monarch, wishing to reward the individual, asked how to repay this individual. The person told the monarch, "Take a chessboard; put 1 cent on the first square, 2 cents on the second, 4 cents on the third, and so on, doubling the amount each time. I wish to be paid the amount that you put on the last (64th) square." This sounded good to the monarch, who agreed. Estimate the amount paid to the individual in terms of powers of 10, in dollars. See problem 51.

53. One of the following is an estimate, in terms of a power of 10 (see problems 51 and 52) of how many grains of sand it would take to fill up a sphere the size of the earth, which has a radius slightly less than 4,000 miles. The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere. Assume that there are about one million grains of sand in a cubic inch, and determine which of the following is the closest estimate.
a. 10^{20} **b.** 10^{30} **c.** 10^{50}
d. 10^{90} **e.** 10^{200} **f.** 10^{500}

54. Using an estimate of one million grains of sand in a cubic inch, estimate the number of grains of sand which it would take to fill the known universe. Assume the known universe is a sphere which is 20 billion light years in radius; a light year is approximately six trillion miles. Determine which of the values in the previous problem is closest to your estimate.

Skill and review

1. Graph $f(x) = \dfrac{x-1}{x^2-4}$.
2. Graph $f(x) = (x-1)(x^2-4)$.
3. Solve $(x-1)(x^2-4) > 0$.
4. Factor $6x^3 + 5x^2 - 2x - 1$.
5. Graph $f(x) = -x^3 + 1$.
6. Solve $x^{2/3} - x^{1/3} - 6 = 0$.

9–2 Logarithmic functions—introduction

The number of binary digits (bits) that a digital computer requires to represent a positive integer N is the smallest integer i such that $2^i \geq N$. Find the number of bits required to represent the integer 35,312.

In this section we define logarithmic functions. We will see that these functions are inverses of exponential functions, and that they prove to be useful in almost any situation in which we use exponential functions. Specifically, these functions will quickly find the value of the exponent i in the section opening problem. As an introduction to these functions we first introduce logarithms.[1]

Logarithms

Consider the statement $2^x = 8$. We can see by inspection that the solution to this statement is 3. In other words, the exponent of the base 2 that "produces" 8 is 3. We also say that the logarithm, to the base 2, of 8 is 3. We use the word logarithm synonymously with the word exponent: A **logarithm** is an exponent. In fact, we continue to use the word logarithm largely for historical reasons. Example 9–2 A illustrates finding logarithms in certain situations—remember that a logarithm is an exponent.

■ ***Example 9–2 A***

Find the unknown logarithm (exponent) in each case.

1. $4^x = 16$
Since $4^2 = 16$ we know that the logarithm x is 2.

2. What is the logarithm of 64 to the base 2?

We want the exponent of the base 2 that produces 64. This is 6 since $2^6 = 64$, so the logarithm, to the base 2, of 64 is 6. ■

A symbolic method was devised to describe the phrase "the logarithm of x to the base b." We write $\log_b x$. Thus, $\mathbf{\log_b x}$ means "the exponent of base b that produces x." For example, $\log_2 16$ is 4, since 4 is the exponent of 2 that produces 16. Example 9–2 B uses this notation.

■ ***Example 9–2 B***

Find the value of the given logarithmic expression.

1. $\log_3 9$
The base is 3; what exponent (logarithm) of 3 gives 9? Since $3^2 = 9$, $\log_3 9 = 2$.

[1]Logarithms were introduced by a Scottish baron, John Napier (1550–1617), in 1614, as a method of simplifying many complex computations. Napier invented the word "logarithm," which means "ratio number." In 1624, Johannes Kepler used the contraction "log," and Henry Briggs, professor of geometry at Oxford, further developed the idea of logarithms in the same year.

2. $3 \log_2 \frac{1}{8}$

We first evaluate $\log_2 \frac{1}{8}$. What exponent of 2 gives $\frac{1}{8}$? We know that $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so the exponent of 2 that gives $\frac{1}{8}$ is -3, and $\log_2 \frac{1}{8} = -3$. Therefore, $3 \log_2 \frac{1}{8} = 3(-3) = -9$.

A general property of logarithms is that since $b^0 = 1$, $\log_b 1 = 0$ for any permissible value of b. Also, since $b^1 = b$, $\log_b b = 1$ for $b > 0$ and $b \neq 1$. This is summarized as follows.

Properties of logarithms

For any base b, $b > 0$ and $b \neq 1$:

Logarithm of one:	$\log_b 1 = 0$
Logarithm of the base:	$\log_b b = 1$

Equivalence of logarithmic and exponential forms

The statement $4 = \log_2 16$ means 4 is the exponent of 2 that produces 16, or $2^4 = 16$. This shows an equivalence between logarithmic and exponential forms. Let us now formalize a definition of the logarithm to the base b of x using this idea of equivalence of forms.

Equivalence of logarithmic and exponential form

$y = \log_b x$ if and only if $b^y = x$, where $b > 0$, $b \neq 1$.

Observe that we put the same restrictions on the value of b as we did for exponential functions in section 9–1.

We have defined logarithms in terms of exponents; $b^y = x$ is an exponential equation and $y = \log_b x$ is a logarithmic equation. Thus, we can change any exponential equation into its equivalent logarithmic form, and vice versa. Notice that *we can rewrite a logarithmic equation as an exponential equation, or vice versa, by moving only the base from one side of the equation to the other.* This rewriting process is illustrated in example 9–2 C.

■ *Example 9–2 C*

1. $3 = \log_{10} 1{,}000$ — The base is 10

$10^3 = 1{,}000$ — Move the base to the other side of the equation; since 10 is a *base* 3 becomes an exponent

2. $\log_m 6 = 18$ — Base is m

$6 = m^{18}$ — Move the base to the other side of the equation

Write each exponential equation as a logarithmic equation.

3. $3^5 = 243$ — The base is 3

$5 = \log_3 243$ — Move the base to the other side; since it must stay a base it must be written $\log_3$

4. $x = 7^2$ Base is 7

$\log_7 x = 2$ Move the base to the other side of the equation

Solving logarithmic equations by rewriting as exponential equations

As illustrated in example 9–2 D, logarithmic equations can often be solved by putting the equation in exponential form.

■ ***Example 9–2 D***

Solve the following equations.

1. $\log_5 x = -2$

$$x = 5^{-2}$$
$$x = \frac{1}{5^2}$$
$$x = \frac{1}{25}$$

2. $\log_4 64 = z$

$$64 = 4^z$$
$$4^3 = 4^z$$
$$z = 3$$

Estimating values of logarithms

Although the values of most logarithms are irrational numbers, it is useful to be able to estimate their values as integers. This uses the following property, which is true when $b > 1$:

$$\text{if } x < y, \text{ then } \log_b x < \log_b y.$$

This is because logarithmic functions with $b > 1$ are increasing functions (as we will see later). In most practical situations involving logarithms, $b > 1$. Example 9–2 E illustrates estimating logarithms using this property.

■ ***Example 9–2 E***

The number of binary digits (bits) that a digital computer requires to represent a positive integer N is the smallest integer greater than or equal to the value $\log_2 N$. Find the number of bits required to represent the integer 5,218.

We need to approximate $\log_2 5{,}218$.

$2^{12} = 4{,}096$ and $2^{13} = 8{,}192$, so $\log_2 4{,}096 = 12$ and $\log_2 8{,}192 = 13$, so $12 < \log_2 5{,}218 < 13$. Therefore, 13 is the number of bits required.

Logarithmic functions

We now define the general logarithmic function.

Logarithmic function

A logarithmic function is a function of the form

$$f(x) = \log_b x, \; b > 0 \text{ and } b \neq 1$$

with

Domain: $\{x \mid x \in R, x > 0\}$

Range: $\{y \mid y \in R\}$.

The logarithmic function to the base b is the inverse function of the exponential function to the base b. Thus, the domain of the logarithmic function is the range of the corresponding exponential function, and the range is the domain of the corresponding exponential function.

The logarithmic and exponential functions are inverses because if an ordered pair (x,y) satisfies the function $f(x) = \log_b x$, its reversal satisfies the function $f(x) = b^x$. For example, the ordered pair (8,3) satisfies the function $f(x) = \log_2 x$, since $3 = \log_2 8$, and the ordered pair (3,8) satisfies the function $f(x) = 2^x$, since $8 = 2^3$. This is a direct result of our definition of logarithms.

The fact that these functions are inverses implies two properties that we now develop. Recall from section 4–5 that if two functions, f and g, are inverses then $f(g(x)) = x$ and $g(f(x)) = x$. Assume that $f(x) = \log_b x$ and $g(x) = b^x$. Then,

$$\begin{aligned} f(g(x)) &= \log_b(g(x)) && f(x) = \log_b x \\ &= \log_b(b^x) && g(x) = b^x \\ &= x && \text{We know } f(g(x)) = x \end{aligned}$$

Also, $g(f(x)) = b^{f(x)} = b^{\log_b x} = x$

Thus, we know that the following two properties are true.

Composition of exponential and logarithmic functions

$$\log_b b^x = x$$
$$b^{\log_b x} = x$$

Example 9–2 F illustrates simplifying certain expressions composed of a logarithmic expression and an exponential expression that are the inverse of each other.

■ Example 9–2 F

Simplify the following, using the properties cited above.

1. $\log_3 3^{10} = 10$ $\quad \log_a a^x = x$

2. $27^{\log_{27} 15} = 15$ $\quad a^{\log_a x} = x$

3. $\log_7 7 = 1$ $\quad \log_b b = 1$

4. $\log_3 1 = 0$ $\quad \log_b 1 = 0$ ■

Mastery points

Can you

- State the definition of the statement $y = \log_b x$?
- Convert between exponential and logarithmic forms of equations?
- Solve simple logarithmic equations?
- Estimate the values of logarithms?
- Use the various properties of logarithms to simplify appropriate expressions?

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Exercise 9–2

Find the unknown logarithm in each case.

1. $3^x = 27$ **2.** $2^y = 128$ **3.** $5^z = 125$ **4.** $2^y = 8$ **5.** $3^x = \frac{1}{27}$
6. $2^n = \frac{1}{8}$ **7.** $2^w = 0.25$ **8.** $4^z = \frac{1}{64}$ **9.** $10^k = 0.1$ **10.** $10^r = 0.001$

Find the value of the expression.

11. $\log_2 8$ **12.** $\log_5 25$ **13.** $\log_4 256$ **14.** $\log_{10} 0.01$
15. $\log_2 \frac{1}{64}$ **16.** $\log_3 \frac{1}{81}$ **17.** $5 \log_3 27$ **18.** $2 \log_{10} 100$
19. $3 \log_5 \frac{1}{125}$ **20.** $2 \log_3 9 + 5 \log_6 36$ **21.** $5(3 \log_2 \frac{1}{8} + 2 \log_{10} 0.1)$ **22.** $3 \log_5 5^2$
23. $\log_4 4^3$ **24.** $(\log_4 4)^3$

Put each logarithmic equation into exponential form.

25. $\log_2 8 = 3$ **26.** $\log_{10} 100 = 2$ **27.** $\log_{10} 0.1 = -1$ **28.** $\log_4 x = 3$
29. $\log_{12}(x + 3) = 2$ **30.** $\log_2 y = 5$ **31.** $\log_3 5 = x + 2$ **32.** $\log_m x = 2y + 1$

Put each exponential equation into logarithmic form.

33. $2^4 = 16$ **34.** $3^4 = 81$ **35.** $x^2 = m + 3$ **36.** $4 = y^{2x-1}$
37. $m^y = x + 1$ **38.** $(x - 1)^3 = 5$ **39.** $(2x - 3)^{x+y} = y + 2$ **40.** $(3x)^2 = 4y$

Solve the following equations for x.

41. $\log_2 x = 4$ **42.** $\log_3 x = 2$ **43.** $\log_2 4 = x$ **44.** $\log_x 64 = 3$
45. $\log_x 64 = 6$ **46.** $\log_{10} x = -2$ **47.** $\log_{16} x = 0.25$ **48.** $\log_x \frac{1}{8} = -3$
49. $\log_x 0.1 = -1$ **50.** $\log_x k = 1$ **51.** $\log_k x = 2$ **52.** $\log_k k^3 = x$

Estimate the values of the following logarithms by stating two consecutive integers that bracket the value, or the value itself if possible.

53. $\log_2 100$ **54.** $\log_3 100$ **55.** $\log_4 100$ **56.** $\log_5 500$ **57.** $\log_2 0.3$ **58.** $\log_2 0.8$

Simplify the following expressions.

59. $\log_2 2^4$ **60.** $10^{\log_{10} 100}$ **61.** $\log_5 5$ **62.** $\log_6 1$ **63.** $\log_4 4^5$ **64.** $2^{\log_2 19}$
65. $\log_{10} 10^{18}$ **66.** $\log_{10} 10^{-4}$ **67.** $\log_a a$ **68.** $m^{\log_m 7}$ **69.** $5 \log_5 1$ **70.** $3 \log_3 3$
71. $-5 \log_2 2$ **72.** $\frac{1}{2} \log_5 5$ **73.** $4^{\log_2 9}$ **74.** $9^{\log_3 4}$ **75.** $6^{\log_6 5^4}$ **76.** $3^{\log_3 1}$

77. The number of binary digits (bits) that a digital computer requires to represent a positive integer N is the smallest integer greater than or equal to the value $\log_2 N$. Find the number of bits required to represent the following integers.
a. 843 **b.** 9,400 **c.** 16,000 **d.** 35,312

78. The cost of a typical \$1 item after a year of inflation at a rate of r percent per year is approximately $V = 2.7^r$. Rewrite this equation in logarithmic form.

79. A relation that relates power I, relative to some fixed power taken as a basic unit, to decibels (d is a measure of sound level) is $\frac{d}{10} = \log_{10} I$. Rewrite this as an exponential equation.

80. The population of a certain bacterial culture is found to fit the relation $P = 5(1.25^t)$, where P is the population after time t. Rewrite this in logarithmic form.

81. The following definition is incorrect; fix the exponential portion of the definition so it is correct. *Definition:* $\log_b x = y$ if and only if $x^y = b$, $b > 0$, $b \neq 1$.

82. Several of the following statements are incorrect. Fix them. For any base b, $b > 0$, $b \neq 1$,
a. $\log_b 1 = 1$ **b.** $\log_b b = 0$
c. $\log_b b^x = x$ **d.** $b^{\log_x b} = x$

Skill and review

1. Rewrite 9^{2x} as a power of 3.
2. Find the inverse function f^{-1} of the function $f(x) = 2 - 3x$ and graph both f and f^{-1}.
3. Solve $2x^6 + 15x^3 - 8 = 0$.
4. Graph $f(x) = x^4 - x$.
5. Solve $\frac{2x - 5}{3} - \frac{3x + 12}{2} = 4$.
6. Solve $2xy = \frac{x + y}{3}$ for y.

9–3 Properties of logarithmic functions

The time t necessary for an amount of money P to grow to an amount A at a fixed interest rate i, compounded daily, is approximated by the relation $t = \frac{1}{i}\log_{2.7}\left(\frac{A}{P}\right)$. Rewrite the right member so that the division $\frac{A}{P}$ is avoided.

In this section we study several properties of logarithms that are used extensively in solving logarithmic equations; they would allow us to do what is asked in this problem.

The one-to-one property of logarithmic functions

Logarithmic functions are one to one (since they have inverse functions). The first of the following properties was stated previously (section 9–1); the second property states the same thing about logarithmic functions.

One-to-one property for exponential functions

If $b^x = b^y$, then $x = y$.

One-to-one property for logarithmic functions

If $\log_b x = \log_b y$, then $x = y$.

The one-to-one property for logarithmic functions is used to solve certain logarithmic equations, as illustrated in example 9–3 A.

■ Example 9–3 A

Solve the equation:

$$\begin{aligned} \log_3 5x &= \log_3(3x + 2) \\ 5x &= 3x + 2 \quad \text{One-to-one property} \\ x &= 1 \end{aligned}$$

■

Three important properties of logarithmic functions

Logarithmic functions have several important algebraic properties. We introduce these properties here, examine why they are true, and see some examples of their use.

Product-to-sum property of logarithms

$\log_b(xy) = \log_b x + \log_b y$, if $x > 0$ and $y > 0$.

Concept

The logarithm of a product is the same as the sum of the logarithms of each factor in the product.

The following shows why this property is true. Let $p = \log_b(xy)$, so $xy = b^p$. Let $q = \log_b x$, so $x = b^q$. Let $r = \log_b y$, so $y = b^r$.

Now we focus on xy:

$xy = x \cdot y$	
$b^p = b^q \cdot b^r$	Replace xy by b^p, x by b^q, y by b^r
$b^p = b^{q+r}$	Property of exponents
$p = q + r$	One-to-one property
$\log_b(xy) = \log_b x + \log_b y$	Replace p by $\log_b(xy)$, q by $\log_b x$, r by $\log_b y$

Example 9–3 B illustrates some ways in which this property is used.

■ *Example 9–3 B*

Apply the product-to-sum property of logarithms in each problem.

1. If $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$, find an approximation for $\log_5 12$.

$\log_5 12 = \log_5(2 \cdot 2 \cdot 3)$	Factor 12 completely
$= \log_5 2 + \log_5 2 + \log_5 3$	Product-to-sum property
$\approx 0.4307 + 0.4307 + 0.6826$	Replace by given values
≈ 1.5440	

2. Solve the logarithmic equation $\log_3 x + \log_3(x + 8) = 2$.

$\log_3 x + \log_3(x + 8) = 2$	
$\log_3(x(x + 8)) = 2$	Product-to-sum property
$x(x + 8) = 3^2$	Rewrite as exponential equation
$x^2 + 8x - 9 = 0$	Multiply in left member
$(x - 1)(x + 9) = 0$	Factor left member
$x - 1 = 0$ or $x + 9 = 0$	Zero product property
$x = 1$ or -9	Solve each linear equation

The domain of any logarithmic function $f(x) = \log_b x$ is the nonnegative real numbers. Therefore, neither $\log_3 x$ nor $\log_3(x + 8)$ is defined for $x = -9$. Either of these undefined expressions means that *we must reject the solution* -9.

Thus, the result is the value 1.

Check for $x = 1$

$\log_3 x + \log_3(x + 8) = 2$	Original equation
$\log_3 1 + \log_3 9 = 2$	Replace x by 1
$0 + 2 = 2$	$\log_3 1 = 0$ and $\log_3 9 = 2$ ■

Just as one adds the logarithms of the factors of a product, one subtracts the logarithms of the factors of a quotient.

Quotient-to-difference property of logarithms

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y, \text{ if } x > 0 \text{ and } y > 0.$$

Concept

The logarithm of a quotient is the same as the difference of the logarithms of the numerator and denominator.

It is left as an exercise to show why this property is true. Example 9–3 C illustrates this property.

■ ***Example 9–3 C***

Use the quotient-to-difference property to solve the logarithmic equation $\log_{10}(x + 99) - \log_{10}x = 2$ for x.

$\log_{10}\dfrac{x+99}{x} = 2$	Quotient-to-difference property
$\dfrac{x+99}{x} = 10^2$	Rewrite as exponential equation
$\dfrac{x+99}{x} = 100$	
$x + 99 = 100x$	Multiply each member by x
$99 = 99x$	Add $-x$ to both members
$1 = x$	Divide both members by 99

Since $\log_{10}(x + 99)$ and $\log_{10}x$ are both defined for $x = 1$ this value will check. ■

The following is one more important property of logarithms.

Exponent-to-coefficient property of logarithms

$$\log_b x^r = r \log_b x, \text{ if } x > 0.$$

Concept

The logarithm of an expression with an exponent is equivalent to the product of that exponent and the logarithm of that expression without the exponent.

The following shows why this is true. Let $m = \log_b x^r$, so $x^r = b^m$. Let $n = \log_b x$, so $b^n = x$.

$b^n = x$	We begin with this statement
$(b^n)^r = x^r$	Raise both sides to power r
$b^{nr} = x^r$	$(a^m)^n = a^{mn}$
$b^{nr} = b^m$	$x^r = b^m$ (because $m = \log_b x^r$)
$nr = m$	Exponential functions are one to one
$r \cdot n = m$	Rewrite order of left member
$r \cdot \log_b x = \log_b x^r$	$n = \log_b x$ and $m = \log_b x^r$

As with the previous properties, this one is often applied in a variety of situations, as illustrated in example 9–3 D.

■ *Example 9–3 D*

Apply the exponent-to-coefficient property of logarithms in each problem.

1. Write $\log_3\dfrac{9x^3y^5}{3z}$ in terms of $\log_3 x$, $\log_3 y$, and $\log_3 z$.

$$\log_3\frac{9x^3y^5}{3z}$$

$= \log_3(9x^3y^5) - \log_3(3z)$	Quotient-to-difference property
$= \log_3 9 + \log_3 x^3 + \log_3 y^5 - (\log_3 3 + \log_3 z)$	Product-to-sum property
$= 2 + 3\log_3 x + 5\log_3 y - 1 - \log_3 z$	Exponent-to-coefficient property
$= 1 + 3\log_3 x + 5\log_3 y - \log_3 z$	

2. Solve the logarithmic equation $\log_2 x^4 = 40$ for x; assume $x > 0$.

$\log_2 x^4 = 40$	
$4 \cdot \log_2 x = 40$	Exponent-to-coefficient property
$\log_2 x = 10$	Divide both members by 4
$x = 2^{10}$	Rewrite as an exponential equation
$x = 1{,}024$	Evaluate 2^{10}

■

Note We assume $x > 0$ in part 2 of example 9–3 D so that the exponent-to-coefficient property would apply. An alternate solution is required without this assumption:

$$\log_2 x^4 = 40$$
$$x^4 = 2^{40}$$
$$x = \pm\sqrt[4]{2^{40}}$$
$$x = \pm 2^{10} = \pm 1{,}024$$

The properties of logarithms and the one-to-one property of exponential functions are summarized here. They should be memorized.

Summary of properties

If $b^x = b^y$, then $x = y$	One-to-one property of exponential functions
If $\log_b x = \log_b y$, then $x = y$	One-to-one property of logarithmic functions
$\log_b(xy) = \log_b x + \log_b y$	Product-to-sum property
$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$	Quotient-to-difference property
$\log_b x^r = r \cdot \log_b x$	Exponent-to-coefficient property

Mastery points

Can you

- Use the algebraic properties of logarithmic functions summarized here to solve certain logarithmic and exponential equations and transform certain logarithmic expressions?

Exercise 9–3

Solve the following logarithmic equations.

1. $\log_2 3x = 3$
2. $\log_4 5x = -2$
3. $\log_2 3x = \log_2 3$
4. $\log_5(1 - 2x) = \log_5 6$
5. $\log_3 5x = \log_3(2x + 1)$
6. $\log_2 3x = \log_2(x - 2)$
7. $\log_3 9 = x$
8. $\log_2 \frac{1}{16} = x$
9. $\log_2(5x - 1) = -4$
10. $\log_2 \dfrac{4}{x} = 3$
11. $\log_2 5x - 1 = -4$
12. $\log_2(-x) = \log_2 \frac{1}{8}$
13. $\log_5(x + 1) = \log_{10} 100$
14. $\log_3(x + 2) = \log_2 \frac{1}{8}$
15. $\log_2 2x + \log_2(x + 1) = 3$
16. $\log_2(x + 1) + \log_2(x - 1) = 4$
17. $\log_2 5 + \log_2(3 - 2x) = \log_2 6$
18. $\log_4 3 = \log_4 x + \log_4(x - 2)$
19. $\log_4 x - \log_4 3 = 2$
20. $\log_4(x + 1) - \log_4 x = 3$
21. $\log_5 x - \log_5 3 = \log_5 2$
22. $\log_5 2 - \log_5 x = \log_5 3$
23. $\log_3 2x = \log_3 2 + \log_3 x$
24. $\log_3 2x = \log_3 2 - \log_3 x$
25. $\log_2 x^4 = 12,\ x > 0$
26. $\log_5 x^3 = \log_5 5^9,\ x > 0$
27. $\log_2(x - 2) + \log_2(x + 3) = \log_2(x^2 - 3x + 2)$
28. $\log_{10}(x - 2) - \log_{10}(x + 3) = \log_{10}(3x + 2)$

Rewrite the following expressions in terms of $\log_a x$, $\log_a y$, and $\log_a z$ for the given value of a.

29. $\log_6(2xy)$
30. $\log_{10}(3xyz)$
31. $\log_4(4xyz) - \log_4 z$
32. $\log_4(2xy) + \log_4(3x)$
33. $\log_3 \dfrac{3xy}{2z}$
34. $\log_2 \dfrac{4x}{y}$
35. $\log_{10} \dfrac{1}{3xyz}$
36. $\log_3 \dfrac{2xyz}{15}$
37. $\log_2 4x^3y^2z^5$
38. $\log_3 9x^2y$
39. $\log_4 \dfrac{8y^4z^3}{x^3}$
40. $\log_{10} \dfrac{3x^{12}y^2z}{1{,}000}$

Assume $\log_a 2 \approx 0.3562$, $\log_a 3 \approx 0.5646$, and $\log_a 5 \approx 0.8271$. Use these values to find approximate values for the following logarithms.

41. $\log_a 6$
42. $\log_a 30$
43. $\log_a 36$
44. $\log_a 10$
45. $\log_a 81$
46. $\log_a 300$
47. $\log_a 0.2$
48. $\log_a 0.5$

49. Referring to the values given above for $\log_a x$, suppose it is also known that $\log_a 14 = 1.3562$. What is a? (Look at the decimal values.)

50. Estimate the value of $\log_2 9 + \log_3 30 + \log_5 20$, to the nearest integer.

51. An equation that occurs when measuring sound levels relative to an initial sound level of 100 is $\alpha = 10 \log_{10}\left(\dfrac{I}{100}\right)$. Rewrite the right member of this equation using the properties of logarithms.

52. The time t necessary for an amount of money P to grow to an amount A at a fixed interest rate i, compounded daily, is approximated by the relation $t = \dfrac{1}{i}\log_{2.7}\left(\dfrac{A}{P}\right)$. Rewrite the right member using the property of the logarithm of a quotient.

53. Use the properties of logarithms to prove that $\log_a \sqrt[n]{x} = \dfrac{\log_a x}{n}$.

54. Prove that $\log_a \dfrac{x}{y} = \log_a x - \log_a y$; use the proof of the fact that $\log_a(xy) = \log_a x + \log_a y$ as a guide.

55. One property of logarithms is $\log_a(xy) = \log_a x + \log_a y$. Show that the following is not a property: $\log_a(x + y) = \log_a x + \log_a y$. Do this by finding values of a, x, and y for which you know all the values and show that the left member of the equation is not equal to the right member for those values.

56. Is the following a property of logarithms (see problem 55)?

$$\log_a(xy) = (\log_a x)(\log_a y)$$

Skill and review

1. If $2^3 < 2^x < 2^4$, what can be said about x?
2. Solve $3^{2x} = 27$ for x.
3. If $\log_x 125 = 3$, what is x?
4. Solve $x^3 + 2x^{3/2} - 3 = 0$.
5. Solve $|2x - 5| = 10$.
6. Graph $f(x) = x^2 + 3x - 5$.

9–4 Values and graphs of logarithmic functions

Probabilistic risk assessment is used to predict the reliability of electronic equipment, aircraft, nuclear power plants, spacecraft, etc. A reliability function R is defined to be $R(t) = e^{-t/\text{MTBF}}$, where t represents time and MTBF means "mean time between failures," the average time it takes for a given piece of equipment to fail. Assuming a MTBF of 1,500 hours for a certain computer, compute the probability that the computer will run for at least 1,000 hours without failure.

In this section we study the mathematical tools we need to answer questions like this one.

Until the 1970s logarithms were used extensively for performing computations involving multiplications, divisions, and extractions of roots. Indeed, this is the very purpose for which Napier created logarithms. Also, the values of logarithms were found using printed tables.

In the 1970s, electronic computing devices made these applications obsolete but made other uses of logarithms more important. For example, we might use a $\boxed{y^x}$ key on a calculator to compute $3.7^{2.1}$. The calculator or computer uses logarithms internally to find the required value. Logarithms are also used extensively in computer science to describe the performance of algorithms. Of course logarithms, like everything else in this text, are also used in more advanced mathematics courses.

The answer to many problems involve the numeric computation of a logarithm. Calculators are programmed to produce the values of logarithms to two bases, 10 and e. The base e is a constant with a value about 2.7. It is discussed later in this section, after we discuss the base 10.

Common logarithms

A scientific electronic calculator is programmed to produce values of **common logarithms.** Common logarithms are logarithms to the base 10. Usually $\log_{10}x$ is abbreviated as simply $\log x$ (the base is assumed to be 10).

Common logarithm

$\log x$ means $\log_{10}x$; it is called the common logarithm of x.

The necessary keystrokes for computing approximations to $\log x$ for a given calculator may differ, but the $\boxed{\log}$ key is practically universal. Some calculators may also require a "function" or "2nd" key.

■ ***Example 9–4 A***

Use a calculator to approximate the values of the following common logarithms. Round the results to 4 decimal places.

	Typical scientific calculator	TI-81
1. log 50 $= 1.6990$	50 $\boxed{\log}$	$\boxed{\text{LOG}}$ 50 $\boxed{\text{ENTER}}$
2. log 0.5 $= -0.3010$	.5 $\boxed{\log}$	$\boxed{\text{LOG}}$ .5 $\boxed{\text{ENTER}}$

Note from example 9–4 A that log 0.5 is negative. The common logarithm of x when $0 < x < 1$ must always be negative. This is because $\log 1 = 0$ and the common logarithm function is an increasing function so if $x < 1$, $\log x < \log 1 = 0$. (We will see the graph of $y = \log x$ shortly. It will confirm that this is an increasing function.)

■ ***Example 9–4 B***

Use a calculator to approximate log 1,230,000,000,000,000 to 5 decimal places.

Since this value is too large to directly enter into a calculator, write it in scientific notation first:

$$1{,}230{,}000{,}000{,}000{,}000 = 1.23 \times 10^{15}.$$

Calculators will accept values in this form, but we will illustrate a more general method, which uses the product-to-sum property.

$$\begin{aligned}
\log(1.23 \times 10^{15}) &= \log 1.23 + \log 10^{15} && \text{Product-to-sum property}\\
&= 0.0899 + 15 \log 10 && \text{Exponent-to-coefficient property}\\
&= 0.0899 + 15 && \log 10 = 1\\
&= 15.0899
\end{aligned}$$

Natural logarithms

Earlier we mentioned that calculators will also calculate logarithms to the base e. The symbol e is used, like π, to represent a certain value. Like π, e is an irrational number; it has been calculated to over 100,000 decimal places, and is approximately 2.718 281 828 459 045 235. The symbol e is credited to Leonard Euler, and first appeared in the year 1727. The origin of e is discussed after example 9–4 G.

Logarithms to the base e are called **natural logarithms,** and $\log_e x$ is often abbreviated[2] as $\ln x$.

Natural logarithm
$\ln x$ means $\log_e x$ and is called the natural logarithm of x.

[2]Irving Stringham used this notation in 1893.

To obtain values of natural logarithms we use a key that is usually marked $\boxed{\ln}$ on a calculator. This is illustrated in example 9–4 C.

■ ***Example 9–4 C***

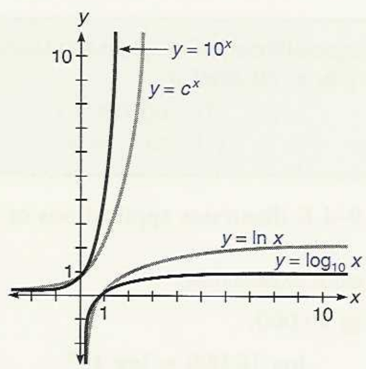
Approximate the natural logarithm; round results to 4 decimal places.

1. ln 100	≈ 4.6052	100 $\boxed{\ln}$	TI-81: $\boxed{\text{LN}}$ 100 $\boxed{\text{ENTER}}$
2. ln 10	≈ 2.3026	10 $\boxed{\ln}$	TI-81: $\boxed{\text{LN}}$ 10 $\boxed{\text{ENTER}}$ ■

Graph of the common and natural logarithm functions

The exponential function to the base e and the natural logarithm function are inverses of each other, since they both use the same base, e. Similarly, the exponential function to the base 10 is the inverse function of the common logarithm function.

The graphs of the common and natural logarithm functions are easily found by reflecting the graphs of the functions $f(x) = 10^x$ and $f(x) = e^x$ about the line $y = x$. This is shown in figure 9–4, which allows us to see the following properties of these functions.



$\boxed{\text{Y=}}$ $\boxed{10^x}$ $\boxed{\text{X|T}}$ $\boxed{\text{ENTER}}$
$\boxed{e^x}$ $\boxed{\text{X|T}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{LOG}}$
$\boxed{\text{X|T}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{LN}}$ $\boxed{\text{X|T}}$
$\boxed{\text{RANGE } -2,8,-2,8}$

Figure 9–4

The domains and ranges for the natural and common logarithm and exponential functions to base 10 and base *e*

Function	Domain	Range
$\log x$, $\ln x$	$\{x \mid x > 0\}$	R
10^x, e^x	R	$\{y \mid y > 0\}$

Observe also that *the common and natural logarithm functions are increasing functions*. Indeed, it can be shown that $f(x) = \log_b x$ is an increasing function if $b > 1$, and a decreasing function if $0 < b < 1$.

There are many instances where we need to find a decimal approximation of 10 or e raised to some power. This is done with a calculator, as shown in example 9–4 D.

■ ***Example 9–4 D***

1. Approximate $10^{3.28}$ to the nearest integer.

Since $10^3 = 1{,}000$ the value should be somewhat more than 1,000. A typical sequence of key strokes is

3.28 [shift] [log] or 3.28 [10^x] — Typical scientific calculator

[2nd] [LOG] 3.28 [ENTER] — TI-81

$10^{3.28} = 1{,}905$ to the nearest integer.

2. Approximate $e^{4.1}$ to the nearest tenth.

Since $e \approx 3$, $e^{4.1} \approx 3^4 = 81$. Using a sequence like

4.1 [shift] [ln] or 4.1 [e^x] — Typical scientific calculator

[2nd] [LN] 3.28 [ENTER] — TI-81

we obtain 60.3, to the nearest tenth. ■

The properties of composition of exponential with logarithmic function, and composition of logarithmic with exponential function, from section 9–2, when put in terms of common and natural logarithms, state:

Composition of exponential/logarithm functions for base 10 and *e*

[1]	$\log 10^x = x$	[3]	$\ln e^x = x$
[2]	$10^{\log x} = x$	[4]	$e^{\ln x} = x$

Example 9–4 E illustrates applications of these properties.

■ ***Example 9–4 E***

Simplify each expression.

1. Find log 10,000.

$$\begin{aligned}\log 10{,}000 &= \log 10^4\\ &= 4 && \log 10^x = x\end{aligned}$$

2. Simplify $\ln(2e^{5x})$.

$$\begin{aligned}\ln(2e^{5x}) &= \ln 2 + \ln e^{5x} && \text{Product-to-sum property}\\ &= \ln 2 + 5x && \ln e^x = x\end{aligned}$$

3. Simplify $e^{\ln(3x)^4}$.

$$\begin{aligned}e^{\ln(3x)^4} &= (3x)^4 && e^{\ln x} = x\\ &= 81x^4\end{aligned}$$ ■

Change-of-base formula

There are many instances where we need to know the value of a logarithm to an arbitrary base. For example, in computer science the bases 2, 8, or 16 are common. In section 9–5 we will also see further instances of the need to compute logarithms to any base.

Suppose then, that we need to compute the quantity $y = \log_b x$, where b is a base for which a calculator is not preprogrammed. Assume that we do have the values to another base, m, available to us. We can develop a formula that will give us the $\log_b x$ in terms using only values of logs to the base m.

Let $y = \log_b x$

$b^y = x$	Rewrite in exponential form
$\log_m b^y = \log_m x$	Take the logarithm to the base m of both members
$y \log_m b = \log_m x$	Exponent-to-coefficient property
$y = \dfrac{\log_m x}{\log_m b}$	Replace y by $\log_b x$

Change-of-base formula

$$\log_b x = \frac{\log_m x}{\log_m b}$$

In practice, we often use $m = 10$, so that the change-of-base property becomes the following.

Change-to-common log formula

$$\log_b x = \frac{\log x}{\log b}$$

This diagram can help us remember the formula: $\log_b x = \dfrac{\log x}{\log b}$. Of course, we can use the base e also; then the property would look like $\log_b x = \dfrac{\ln x}{\ln b}$. Example 9–4 F illustrates the change-of-base formula.

■ ***Example 9–4 F***

Use the change-to-common log formula to compute $\log_7 100$ to four decimal places.

$$\log_7 100 = \frac{\log 100}{\log 7} \approx \frac{2}{0.8451} \approx 2.3666$$

100 $\boxed{\text{log}}$ $\boxed{\div}$ 7 $\boxed{\text{log}}$ $\boxed{=}$	Typical scientific calculator
$\boxed{\text{LOG}}$ 100 $\boxed{\div}$ $\boxed{\text{LOG}}$ 7 $\boxed{\text{ENTER}}$	TI-81

It is a good idea to check this value in the following way:

Since $7^2 = 49$ and $7^3 = 343$, we know $2 < \log_7 100 < 3$. ■

Graphs of logarithmic functions

In section 4–5 we observed that the graphs of a function and its inverse function are symmetric about the line $y = x$. Since the logarithmic and exponential functions are inverses their graphs are mirror images about this line. This is illustrated in figure 9–5; part a shows the case for $b < 1$, and part b for $b > 1$.

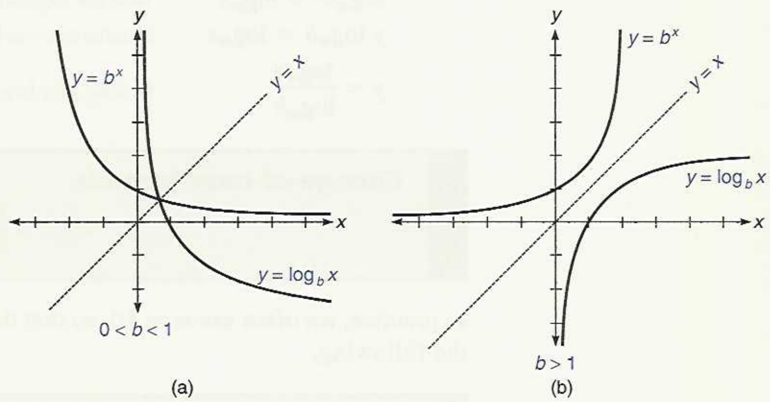


Figure 9–5

Since we already know how to graph exponential functions (from section 9–1), and since a logarithmic function is the inverse of some exponential function, we can use exponential functions to graph logarithmic functions in the following way.

> **To graph a logarithmic function**
> 1. Replace $f(x)$ by y.
> 2. Replace every x by y and every y by x.
> 3. Solve for y. (This is the inverse of the logarithmic function.)
> 4. Graph this equation. (This is the inverse of the desired graph.)
> 5. Reflect the result about the line $y = x$ to obtain the desired graph.

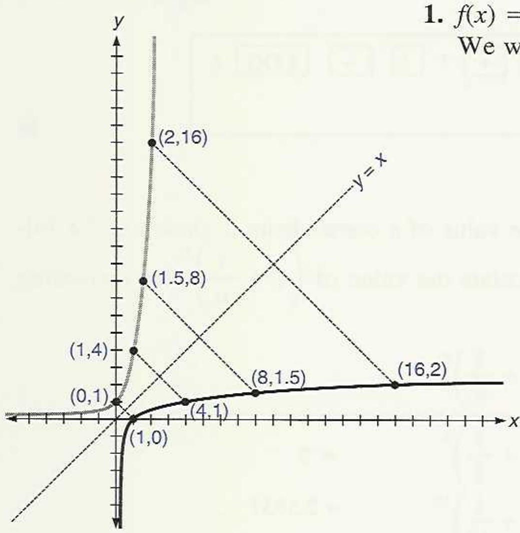
This method of graphing logarithmic functions has the advantage that we do not have to memorize any more basic graphs.

Another method for graphing these functions is to memorize the basic graph of $y = \log_b x$ and use vertical and horizontal translations and vertical scaling, as covered in section 3–4.

Of course the graphing calculator or computer can be used to obtain these graphs also. The essential information for the TI-81 is shown.

■ ***Example 9–4 G***

Graph the following logarithmic functions. State the value of any intercepts.



1. $f(x) = \log_4 x$

We write as $y = \log_4 x$, then find the inverse function:

$x = \log_4 y$ — Exchange x and y

$y = 4^x$ — Solve for y

Graph the function $y = 4^x$ first, then reflect the result about the line $y = x$. This is shown in the figure.

x-intercept:

$0 = \log_4 x$ — Replace $f(x)$ by 0

$4^0 = x$

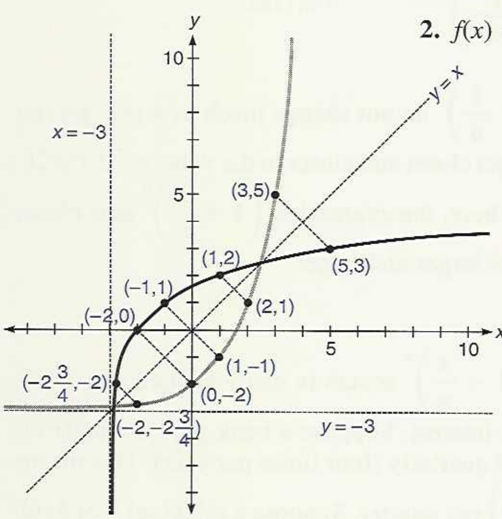
$1 = x$

Points for $y = 4^x$.

x	4^x
−1	0.25
1	4
1.5	8
2	16

Use the change-of-base formula to rewrite $f(x) = \log_4 x$ as $\dfrac{\log x}{\log 4}$.

$\boxed{\text{Y=}}$ $\boxed{\text{LOG}}$ $\boxed{\text{X|T}}$ $\boxed{\div}$ $\boxed{\text{LOG}}$ 4 $\boxed{\text{RANGE } -1,17,-2,3}$



2. $f(x) = \log_2(x + 3)$

We find the inverse:

$y = \log_2(x + 3)$ — $y = f(x)$

$x = \log_2(y + 3)$ — Exchange x and y

$y + 3 = 2^x$ — Write in exponential form

$y = 2^x - 3$ — Solve for y

Graph $y = 2^x - 3$, the inverse function, then reflect about the line $y = x$, as shown.

y-intercept:

$y = \log_2(0 + 3)$ — Replace x by 0 in $y = \log_2(x + 3)$

$y = \log_2 3$ — y-intercept is $(0, \log_2 3)$

x-intercept:

$0 = \log_2(x + 3)$ — Replace y by 0 in $y = \log_2(x + 3)$

$2^0 = x + 3$

$-2 = x$ — x-intercept is $(-2, 0)$

Points for $y = 2^x - 3$.

x	$2^x - 3$
−2	−2.75
−1	−2.5
1	−1
2	1
3	5

Use the change-of-base formula to rewrite f as $f(x) = \dfrac{\log(x + 3)}{\log 2}$.

[Y=] [LOG] [(] [X|T] [+] 3 [)] [÷] [LOG] 2
[RANGE −4,10,−4,10]

The number e

One way to appreciate where the value of e comes from is shown in the following sequence, where we calculate the value of $\left(1 + \dfrac{1}{n}\right)^n$ for increasing values of n.

n	$\left(1 + \dfrac{1}{n}\right)^n$	
1	$\left(1 + \dfrac{1}{1}\right)^1$	$= 2$
10	$\left(1 + \dfrac{1}{10}\right)^{10}$	$\approx 2.5937\ldots$
100	$\left(1 + \dfrac{1}{100}\right)^{100}$	$\approx 2.7048\ldots$
1,000	$\left(1 + \dfrac{1}{1{,}000}\right)^{1{,}000}$	$\approx 2.7169\ldots$
10,000	$\left(1 + \dfrac{1}{10{,}000}\right)^{10{,}000}$	$\approx 2.7181\ldots$

As we can see the values of $\left(1 + \dfrac{1}{n}\right)^n$ do not change much as n gets greater and greater.[3] In fact, these values get closer and closer to the value $e \approx 2.71828$.

Although we cannot prove it here, the expression $\left(1 + \dfrac{x}{n}\right)^n$ gets closer and closer to the value e^x as n gets larger and larger.

Compound interest

It turns out that the expression $\left(1 + \dfrac{x}{n}\right)^n$ occurs in many applied situations. For example, consider compound interest. Suppose a bank pays a simple interest rate of $i = 8\%$, compounded quarterly (four times per year). This means that the bank really pays 2% $\left(\dfrac{8\%}{4}\right)$ per quarter. Suppose a principal P of \$100 is deposited. The table shows how much is in the account after each quarter, using the fact that to compute total amount at the end of a quarter, increase the previous amount by 2%. This is accomplished in one step by multiplying the previous amount by 102% or 1.02.

[3]The TI-81 handles this calculation up to n = one billion (1 [EE] 12), but fails at $n = 10$ billion (1 [EE] 13). Calculators do have limitations.

End of quarter	Computation	Amount	Computations to date
1	\$100(1.02)	\$102	$\$100(1.02)^1$
2	\$102(1.02)	\$104.04	$\$100(1.02)^2$
3	\$104.04(1.02)	\$106.12	$\$100(1.02)^3$
4	\$106.12(1.02)	\$108.24	$\$100(1.02)^4$
5	\$108.24(1.02)	\$110.40	$\$100(1.02)^5$
...	...	...	...
m	...	...	$\$100(1.02)^m$

Thus to compute the present value A in a bank account paying a yearly simple interest rate i compounded n times per year after t years, on an initial deposit, P (the principal) one computes

$$[1] \quad A = P\left(1 + \frac{i}{n}\right)^{nt}$$

For one year $t = 1$, and the amount is $A = P\left(1 + \frac{i}{n}\right)^n$. As n, the number of compounding periods, increases, this quantity gets closer and closer to the value given by $A = Pe^i$, which defines the amount paid in an account (after one year) in which the interest is said to be compounded continuously. After time t, in years, the amount in an account on which the interest at rate i is compounded continuously is

$$[2] \quad A = Pe^{it}$$

Example 9–4 H illustrates both of these formulas.

■ ***Example 9–4 H***

1. A bank account pays 10% simple interest, compounded quarterly. What will be the value of a deposit of \$1,000 after two years?

Use formula [1] with $P = 1{,}000$, $i = 10\% = 0.1$, $n = 4$, and $t = 2$:

$$\begin{aligned} A &= 1{,}000\left(1 + \frac{0.1}{4}\right)^{4(2)} \\ &= 1{,}000(1.025)^8 \\ &\approx 1{,}000(1.21840) \\ &\approx 1{,}218.40 \end{aligned}$$

Thus, \$1,000 grows to \$1,218.40 after two years if interest is compounded quarterly.

2. Assume the money of part 1 of this example is deposited in an account in which the interest is compounded continuously. What is the amount after two years?

Use formula [2] with $i = 0.1$ and $t = 2$:

$$\begin{aligned} A &= 1{,}000e^{0.1(2)} \\ &= 1{,}000e^{0.2} \\ &\approx 1{,}000(1.22140) \approx 1{,}221.40 \end{aligned}$$

Thus, \$1,000 grows to \$1,221.40 after two years if interest is compounded continuously. ■

Mastery points

Can you

- Use a calculator to compute the common or natural logarithm of any positive real number?
- Use the change-to-common log formula to compute the value of a logarithm to any base?
- Sketch the graphs of the natural and common logarithm functions and state their domains and ranges?
- Graph logarithmic equations?
- Use the formulas $A = P\left(1 + \frac{i}{n}\right)^{nt}$ and $A = Pe^{it}$ to compute compound interest?

Exercise 9–4

Use a calculator to approximate the values of the following logarithms. Round the results to 4 decimal places.

1. log 52
2. log 17
3. log 2.55
4. log 190
5. log 10.6
6. log 2,500
7. log 0.85
8. log 0.003
9. log 8,720
10. ln 52
11. ln 17
12. ln 2.55
13. ln 190
14. ln 10.6
15. ln 2,500
16. ln 0.85
17. ln 0.003
18. ln 8,720
19. log 7,920,000,000,000,000
20. log 2,003,400,000,000,000,000,000,000
21. log 90,000,000,000,000,000,000
22. log 718,420,000,000,000
23. log 0.000 000 000 000 000 2
24. log 0.000 000 000 000 000 009 129
25. log 0.000 000 000 120 004
26. Compute the common logarithm of Avogadro's number (from chemistry), to four decimal places; Avogadro's number is 6.024×10^{23}.
27. Approximate the common logarithm of Planck's constant (from physics), 6.63×10^{-27}, to four decimal places.
28. The speed of light in a vacuum is about 2.99776×10^{10} centimeters per second. Compute the common logarithm of this value to four decimal places.
29. Assume the age of the known universe to be 18 billion years; compute the common logarithm of this value, to four decimal places.

Approximate the given logarithm; round to 4 decimal places.

30. $\log_5 19$
31. $\log_{20} 2{,}000$
32. $\log_2 3.89$
33. $\log_8 0.78$
34. $\log_{0.25} 8$
35. $\log_3 0.95$
36. Sketch the graph of the common logarithm function.
37. Sketch the graph of the natural logarithm function.
38. State the domain and range of the common and natural logarithm functions.

Graph the following logarithmic functions. State the value of all intercepts.

39. $f(x) = \log_3 x$
40. $f(x) = \log_4(x - 2)$
41. $f(x) = \log_2(x - 1)$
42. $f(x) = \log_3 2x$
43. $f(x) = \log_4 x - 1$
44. $f(x) = \log_2 x + 2$
45. $f(x) = \log_2 3x$
46. $f(x) = \log_3(-x)$

Approximate the following values to 2 decimal places.

47. $10^{2.9}$
48. $10^{4.82}$
49. $10^{-0.33}$
50. $e^{3.1}$
51. $e^{1.85}$
52. $e^{10.6}$
53. $e^{4.8}$
54. $10^{3.02}$
55. $10^{2/3}$

Simplify the following expressions.

56. $\log 1{,}000$ **57.** $\ln 2e^{3x}$ **58.** $\ln e^{(2x+1)}$ **59.** $\log 10^{(4-3x)}$ **60.** $10^{\log 100}$ **61.** $e^{\ln 100}$
62. $e^{\ln x^2}$ **63.** $10^{\log (x-1)^2}$ **64.** $e^{\ln 3x} + \ln e^{3x}$ **65.** $\ln 5e^x$ **66.** $10^{\log \sqrt{2}}$

67. \$1,800 is deposited in a bank account that computes its interest continuously. The simple interest rate is 8.5% per year. Use the formula $A = Pe^{it}$ to compute the amount in the account after $5\frac{1}{2}$ years.

68. Find the amount of money on deposit in a bank account after 18 months if the initial deposit was \$5,000 and the simple yearly interest rate is 11.5% compounded continuously. See problem 67.

69. Use the formula $A = P\left(1 + \dfrac{i}{n}\right)^{nt}$ to find the value, after 2 years, of an account in which \$1,000 was deposited at an interest rate of 8% per year if the interest is compounded monthly.

70. Use the formula $A = P\left(1 + \dfrac{i}{n}\right)^{nt}$ to find the value after 18 months of an account in which \$1,800 was deposited at an interest rate of 6.5% per year, if the interest is compounded monthly.

71. The logarithms created by John Napier in the seventeenth century were neither common nor natural logarithms. According to Howard Eves in his book *Great Moments in Mathematics Before 1650,* Napier's values could be found by the function
Nap $\log x = 10^7 \log_{1/e}\left(\dfrac{x}{10^7}\right)$.
Show that
Nap $\log x = 10^7(7 \ln 10 - \ln x)$.

72. Referring to problem 71, compute the values that Napier would have obtained for his system of logarithms for the following values of x.
a. 1 **b.** 10 **c.** 100 **d.** one million
Leave answers in scientific notation with four-digit accuracy.

73. The Weber-Fechner law, from psychology, states that sound loudness S is given by $S = k \log\left(\dfrac{I}{I_0}\right)$, where I is the intensity of sound compared to an initial reference intensity, I_0. Assuming that $k = 12$, find S if I is 6 times the value of I_0. Round to one decimal place.

74. The Smith chart is used in electronics to study the performance of antennas. It uses the property that $\log n$ and $\log \dfrac{1}{n}$ are equal distances from zero. Show that this is true; that is, show that $|\log n| = \left|\log \dfrac{1}{n}\right|$ for all $n > 0$.

75. The surge impedance Z_0, in ohms, in a two-wire conductor is given by the relation $Z_0 = k \log \dfrac{b}{a}$, where a is the radius of the wires and b is the distance between the centers of the wires. If $k = 276$, find Z_0 for wire of radius $\frac{1}{8}$ inches, with centers separated by a distance of $\frac{3}{4}$ inches. Round to the nearest unit.

76. Probabilistic risk assessment is used to predict the reliability of pieces of electronic equipment, aircraft, nuclear power plants, spacecraft, etc. A reliability function R is defined to be $R(t) = e^{-t/\mathrm{MTBF}}$, where t represents time and MTBF means mean time between failures, the average time it takes for a given piece of equipment to fail. Assuming a MTBF of 1,500 hours for a certain computer, compute R for 1,000 hours (the probability that the computer will run for at least 1,000 hours without failure). Round to two decimal places.

77. In designing a heating/cooling system that depends on water flowing through a pipe buried in the earth, one uses the formula $Q = 0.07L\left(\dfrac{T_{\text{in}} - T_{\text{out}}}{\log \dfrac{T_{\text{earth}} - T_{\text{in}}}{T_{\text{earth}} - T_{\text{out}}}}\right)$, where Q is heat transfer in BTU/hour, T is the temperature at the pipe inlet (in) and outlet (out) and of the earth, and L is the pipe length. Assuming the temperature of the earth to be 54°, the temperature at the inlet to be 30°, and at the outlet to be 42°, with a pipe length of 80 feet, find Q.

78. An oblate spheroid is similar in appearance to an egg; the earth has the shape of an oblate spheroid. The surface area S of such an object is given by $S = 2\pi a^2 + \pi\dfrac{b^2}{\varepsilon}\ln\left(\dfrac{1+\varepsilon}{1-\varepsilon}\right)$, where a, b, and ε (eccentricity) are parameters which describe the spheroid. Find S to the nearest tenth if $a = 14$ inches, $b = 8$ inches, and $\varepsilon = \frac{1}{4}$.

79. In medicine, the formula

$$D_{L_{CO}} = \frac{V_A}{(P_B - 47)(t_2 - t_1)} \cdot \log \frac{F_{A_{CO_{t_1}}}}{F_{A_{CO_{t_2}}}}$$ is part of computing carbon monoxide (CO) diffusing capacity ($D_{L_{CO}}$) across the alveolocapillary membrane, where V_A = alveolar volume, P_B = barometric pressure, $t_2 - t_1$ = time interval of measurement, $F_{A_{CO_{t_1}}}$ = fraction of CO in alveolar gas before diffusion, and $F_{A_{CO_{t_2}}}$ = fraction of CO in alveolar gas at end of diffusion. Find $D_{L_{CO}}$ if $V_A = 40$ centiliters, $P_B = 52$, $t_2 - t_1 = 10$ seconds, $F_{A_{CO_{t_1}}} = 0.01$, and $F_{A_{CO_{t_2}}} = 0.004$. The units will be centiliters per second.

Skill and review

1. Solve $2x^2 - 9x + 4 = 0$.
2. Solve $2x^4 - 9x^2 + 4 = 0$.
3. Solve $2x - 9\sqrt{x} + 4 = 0$.
4. Solve $2(x - 3)^2 - 9(x - 3) + 4 = 0$.
5. Rewrite $\log_a \frac{2x^4}{3y^3z}$ in terms of logarithms to the base a.
6. Solve $\log_2 x = -3$.
7. Graph $f(x) = \log_3(x - 1)$.
8. Solve $\frac{x^2 - 4}{x^2 - 1} > 2$.

9–5 Solving logarithmic and exponential equations/applications

The Richter scale was invented in 1935 by Charles F. Richter to measure the intensity of earthquakes. Each number on the scale represents an earthquake 10 times stronger than one of the next lower magnitude. For example, an earthquake of 5 on the Richter scale is 10 times stronger than one of measure 4 on the Richter scale. Suppose one earthquake measures 4.5 on the Richter scale, and a second measures 6.2. To the nearest unit, how many times stronger is the second earthquake than the first?

In this section we study ways to solve problems like this, which can be described using logarithmic and exponential equations. We first look at the various techniques available to us to solve these types of equations, and then we see a few of the many places where these equations arise.

Equation solving techniques

In this section we introduce more techniques for solving logarithmic and exponential equations.

Some equations can be solved by taking the logarithm of both members of the equation. Although we will use common logarithms, the procedure and result is the same using natural logarithms. Example 9–5 A illustrates this method of solving equations.

■ *Example 9–5 A*

Solve the equation for x; also find a decimal approximation to the answer to the nearest 0.1.

$3^{x+1} = 2^{3x-1}$

$\log 3^{x+1} = \log 2^{3x-1}$ — Take the common logarithm of both members

$(x + 1)\log 3 = (3x - 1)\log 2$ — Exponent-to-coefficient property

$x \log 3 + \log 3 = 3x \log 2 - \log 2$	Perform the indicated multiplications
$x \log 3 - 3x \log 2 = -\log 2 - \log 3$	Put x terms all in one member
$x(\log 3 - 3 \log 2) = -(\log 2 + \log 3)$	Factor x in the left member
$x = -\dfrac{\log 2 + \log 3}{\log 3 - 3 \log 2}$	Divide each member by $\log 3 - 3 \log 2$
$x \approx 1.8$	Approximate value

2 [log] [+] 3 [log] [=] [÷] [(] 3 [log] [−] 3 [×] 2 [log] [)] [=] [+/−]

TI-81: [(−)] [(] [(] [LOG] 2 [+] [LOG] 3 [)] [÷] [(] [LOG] 3 [−] 3 [LOG] 2 [)] [)] [ENTER] ■

An expression of the form $(\log x)^2$ is often written as $\log^2 x$ for convenience. It is important to distinguish between $\log^2 x$ and $\log x^2$. $\log^2 x$ means $(\log x)^2$, whereas $\log x^2$ means $\log(x^2)$.

The expression $\log^2 x$ means to evaluate the logarithm, then square that value. $\log x^2$ means square x, then evaluate the logarithm. For example, if $x = 100$,

$$\log^3 100 = (\log 100)^3 = 2^3 = 8$$
$$\log 100^3 = 3 \log 100 = 3(2) = 6.$$

Example 9–5 B shows that the technique of substitution for expression can help in some cases (section 1–3).

■ *Example 9–5 B*

1. Under certain conditions the equation $y = \frac{1}{2}(e^x + e^{-x})$ describes the shape of a cable (such as a phone line) hanging between two poles. Solve this equation for x when y has the value 3.

$$y = \tfrac{1}{2}(e^x + e^{-x})$$
$$3 = \tfrac{1}{2}(e^x + e^{-x})$$
$$6 = e^x + e^{-x}$$
$$6 = e^x + \frac{1}{e^x}$$

Let $u = e^x$

$$6 = u + \frac{1}{u}$$
$$6u = u^2 + 1$$
$$u^2 - 6u + 1 = 0$$
$$u = 3 \pm 2\sqrt{2}$$
$$e^x = 3 \pm 2\sqrt{2}$$
$$x = \ln(3 \pm 2\sqrt{2})$$
$$x \approx 1.76 \text{ or } -1.76$$

2. Solve $(\log x)(2 \log x + 1) = 6$

$$2 \log^2 x + \log x - 6 = 0$$
$$2u^2 + u - 6 = 0$$
$$(2u - 3)(u + 2) = 0$$
$$2u - 3 = 0 \text{ or } u + 2 = 0$$
$$u = \tfrac{3}{2} \text{ or } u = -2$$
$$\log x = \tfrac{3}{2} \text{ or } \log x = -2$$
$$10^{3/2} = x \text{ or } 10^{-2} = x$$
$$x = 10\sqrt{10} \text{ or } x = \frac{1}{100}$$
$$x \approx 31.6 \text{ or } 0.01$$

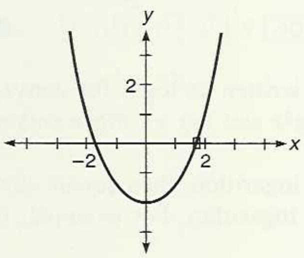
■

Graphing calculators and computers provide a powerful tool for estimating the values of solutions to equations that can be expressed in terms of one variable. Example 9–5 C reviews one way to solve equations with graphing calculators. This was shown in more detail in section 4–3.

■ *Example 9–5 C*

Solve $3 = \frac{1}{2}(e^x + e^{-x})$ graphically. (Part 1 of example 9–5 B.)

Graph the function $y = \frac{1}{2}(e^x + e^{-x}) - 3$. The solutions to the original problem are the zeros of this graph.



The graph is as shown. Use TRACE to position the cursor as shown, then $\boxed{\text{PRGM}}$ NEWTON (section 4–3) to obtain a highly accurate value for the zero. Doing this displays the value 1.762747174. Analysis of the TI-81 program NEWTON will show that it delivers a result that differs from the actual value by less than one part in 10 billion. That is, $\left|\frac{\text{actual } x - \text{estimated } x}{\text{actual } x}\right| \le 10^{-10}$.

It can be seen from the graph that the other zero is the negative of the first, so it is not necessary to compute it. (Formally, $y = \frac{1}{2}(e^x + e^{-x}) - 3$ is an even function (section 3–5), which means its graph is symmetric about the y-axis.) ■

It is important to note that *the graphical method shown in example 9–5 C is of limited usefulness for many of the equations in this section.* This is because the graphs are not easy to view in their entirety and because there are mathematical limitations to all calculating devices. For example, the answer to $\log(\log x) = 2$ is 10^{100}, a value that is beyond the capability of most calculators and computers. The answers to $\log\sqrt{x} = \sqrt{\log x}$ are 1 or 10,000. Whatever graph is used will not show both solutions. We must reset the range to discover them both. This is an impractical procedure in these extreme cases.

Applications

Logarithmic and exponential equations appear in many business and scientific applications.

Compound interest applications

The formula $A = Pe^{it}$ relates present amount A, principal P, and time t in years in an account or loan paying (or demanding) simple interest rate i, compounded continuously. (This was introduced in the previous section.) Example 9–5 D illustrates this formula.

■ *Example 9–5 D*

An account pays 12% interest compounded continuously. How long would it take $1,000 to double in value in this account?

$P = 1{,}000$, $i = 0.12$, and $A = 2{,}000$; we want t.

$A = Pe^{it}$	
$2{,}000 = 1{,}000e^{0.12t}$	Replace variables with the given values
$2 = e^{0.12t}$	Divide each member by 1,000
$\ln 2 = \ln e^{0.12t}$	Take the natural logarithm of both members
$\ln 2 = 0.12t$	$\ln e^x = x$
$\dfrac{\ln 2}{0.12} = t$	Divide both members by 0.12; this is the exact value of t
$5.78 \approx t$	2 $\boxed{\ln}$ $\boxed{\div}$ 0.12 $\boxed{=}$

Thus, it takes about 5.8 years for the $1,000 to attain the value $2,000. ■

Growth and decay applications

A large class of problems are called **growth and decay problems.** In a situation where the rate of growth or decay of something is a constant proportion of the amount present, exponential equations describe the situation.

> **Growth and decay**
>
> If a quantity varies continuously at a certain rate r then the quantity q after time t starting with an original quantity q_0, is given by the growth/decay equation
>
> $$q = q_0e^{rt}$$
>
> When $r > 0$ we have growth, and when $r < 0$ we have decay.

Example 9–5 E illustrates a growth and decay situation.

■ *Example 9–5 E*

How long will it take the earth's population to double if it grows continuously at the rate of 2.7 percent per year?

If an original population q_0 doubles, then at that time $q = 2q_0$. We are also told that $r = 0.027$.

$q = q_0e^{rt}$	Growth/decay equation
$2q_0 = q_0e^{0.027t}$	Replace variables by given values
$2 = e^{0.027t}$	Divide both members by q_0.
$\ln 2 = \ln e^{0.027t}$	Take the natural logarithm of both members
$\ln 2 = 0.027t$	$\ln e^x = x$
$t = \dfrac{\ln 2}{0.027}$	Divide both members by 0.027
$t \approx 25.7$	

Thus, at 2.7% annual growth the earth's population will double in about 26 years. ■

Two additional applications are illustrated in example 9–5 F.

■ *Example 9–5 F* Solve the following applications.

Decibel levels

The intensity of sound is described in units called decibels. One decibel = 0.1 bel, named after Alexander Graham Bell. The **intensity level,** α (alpha) of a sound is defined by the equation $\alpha = 10 \log \frac{I}{I_0}$, where the units are decibels, where I is the present intensity (power) of a sound, and I_0 is an initial or baseline intensity.

1. Find the change in power that would result in a 1 decibel increase in the intensity level of a sound; that is, find the value of I for which $\alpha = 1$.

$$\alpha = 10 \log \frac{I}{I_0}$$

$$1 = 10 \log \frac{I}{I_0}$$ Replace α by 1

$$0.1 = \log \frac{I}{I_0}$$ Divide each member by 10

$$10^{0.1} = \frac{I}{I_0}$$ Rewrite in exponential form

$$10^{0.1} I_0 = I$$ Multiply both members by I_0; exact value of I

$$1.259 I_0 = I$$ $10^{0.1} \approx 1.259$

Thus, the intensity must be about 1.26 times that of the initial intensity to get a 1 decibel increase.

Resistance-capacitance (RC) time constants

In electronic circuit theory, an **RC time constant** describes how long it will take a capacitor to take on about 63% of its electrical charge. For a given fraction q of full charge, $0 < q < 1$, the time t, in RC time constants, is given by the relation $q = 1 - e^{-t}$. This type of circuit can be used in applications like the delay in automobile windshield wipers, or to turn off an automobile's headlights after a time delay.

2. How many RC time constants are necessary to charge a capacitor to 50% of its full capacity; that is, for what value of t is $q = 0.50$?

$$q = 1 - e^{-t}$$

$$0.5 = 1 - e^{-t}$$ Replace q by 0.5

$$e^{-t} = 0.5$$

$$(e^{-t})^{-1} = 0.5^{-1}$$ Raise both members to the -1 power to change the sign of the exponent of e

$$e^{t} = 2$$ $(e^{-t})^{-1} = e^{t}$; $0.5^{-1} = (\frac{1}{2})^{-1} = 2$

$$\ln e^{t} \approx \ln 2$$ Take the natural logarithm of both members

$$t \approx \ln 2$$ $\ln e^{x} = x$

$t \approx 0.69$, to the nearest hundredth

Thus, it takes about 0.69 time constants for a capacitor to be charged to 50% of its full capacity. ■

Obtaining growth and decay formulas from measurements (optional)

In the laboratory, we often have several measurements with which to obtain a growth/decay formula that describes the situation. Example 9–5 G illustrates how to obtain the correct values of q_0, r, and t.

■ ***Example 9–5 G***

A population of bacteria is assumed to be growing continuously at a fixed rate. There are initially 28 μg (micrograms) of the bacteria; 3 hours later there are 40 μg.

a. How many micrograms of bacteria will there be after 6 hours?
b. When will there be 100 μg of bacteria?

$$
\begin{aligned}
&q_0 = 28 \text{ and } q = 40 \text{ when } t = 3\\
&q = q_0e^{rt} \qquad \text{Basic growth/decay formula}\\
&40 = 28e^{3r}\\
&\tfrac{10}{7} = e^{3r}\\
&\ln \tfrac{10}{7} = \ln e^{3r}\\
&\ln \tfrac{10}{7} = 3r\\
&r = \tfrac{1}{3} \ln \tfrac{10}{7}\\
&r \approx 0.1189
\end{aligned}
$$

Thus, the equation that describes the growth of this population is $q = 28e^{0.1189t}$ or $q = 28(1.1262^t)$.

Note Since $e^{\ln(10/7)} = \frac{10}{7}$, the equation is also $q = 28(\frac{10}{7})^{t/3}$.

a. Now find q when $t = 6$: $q = 28(1.1262^6) = 57.1$ μg.
b. We now find when the population will be $q = 100$.

$$
\begin{aligned}
100 &= 28e^{0.1189t}\\
\tfrac{25}{7} &= e^{0.1189t}\\
\ln \tfrac{25}{7} &= 0.1189t\\
t &= \frac{\ln \frac{25}{7}}{0.1189} \approx 10.7
\end{aligned}
$$

Thus, the population will be 100 μg after 10.7 hours of growth (or 7.7 hours after it reaches 40 μg). ■

The TI-81 calculator is preprogrammed to help with problems like that in example 9–5 G. Through a feature called ExpReg (exponential regression) the calculator can find the equations required directly. This is illustrated in example 9–5 H.

Example 9–5 H A population of bacteria is assumed to be growing continuously at a fixed rate. There are initially 28 μg (micrograms) of the bacteria; 3 hours later there are 40 μg.

a. How many micrograms of bacteria will there be after 6 hours?

b. When will there be 100 μg of bacteria?

We have two (x,y) or (time, quantity) pairs: (0,28) and (3,40). We want the value of x in the ordered pair $(x,100)$.

We obtain the exponential equation that "interpolates" (passes through) the first two ordered pairs. Proceed as follows:

[STAT] DATA 2 [ENTER] — Clear out any old data

[STAT] DATA 1 — Edit data

0 [ENTER] 28 [ENTER] 3 [ENTER] 40

[STAT] 4 [ENTER] — Select exponential regression

$a = 28$, $b = 1.12624788$

The equation is $q = 28(1.12624788^t)$.

a. 6 [STO▸] [X|T] [ENTER] — Put 6 in x

[VARS] LR 4 [ENTER] — Evaluate $28(1.12624788^x)$ with $x = 6$

The result is 57.1 μg.

Note We can easily graph $y = 28(1.12624788^x)$ as follows

[Y=] [CLEAR] [VARS] LR 4 — Enter the equation obained above

[RANGE −10,10,−1,100] YSCL=5 — Use these settings for the graph

b. The best way to solve part b is to realize that a logarithmic equation is the inverse of an exponential equation. Recall that the ordered pairs of a function are reversed in its inverse (section 4–5). We reverse the ordered pairs above to obtain (28,0) and (40,3). Then, we want y in the ordered pair $(100,y)$. Instead of doing exponential regression, we do logarithmic regression. This gives the values of a and b in the equation $y = b \ln x + a$.

[STAT] DATA 2 [ENTER] — Clear out any old data

[STAT] DATA 1 — Edit data

28 [ENTER] 0 [ENTER] 40 [ENTER] 3

[STAT] 3 [ENTER] — Select logarithmic regression

$a = -28.02723797$, $b = 8.411019757$, which corresponds to $y = 8.411 \ln x - 28.03$.

100 [STO▸] [X|T] [ENTER] — Put 100 in x

[VARS] LR 4 [ENTER] — Evaluate $8.411 \ln x - 28.03$ with $x = 100$

The result is 10.7 hours. ■

Mastery points

Can you

- Solve certain exponential and logarithmic equations?
- Solve certain applications problems involving exponential and logarithmic equations?

Exercise 9–5

Solve for x.

1. $8^x = 32^{3-2x}$ **2.** $5^{3x} = 25^x$ **3.** $27^{4x} = 9^{x+2}$
4. $(\sqrt{3})^x = 9^{x-2}$ **5.** $(\sqrt{8})^{2x-2} = 4^{3x}$ **6.** $8^{x+1} = 4^{2-x}$
7. $\log(x - 1) + \log(x + 3) = \log 4$ **8.** $\log(x - 1) + \log(2x) = \log 4$ **9.** $\log(x + 1) - \log(x - 3) = \log 4$
10. $\log(x + 1) - \log(2x) = \log 4$ **11.** $\log(x - 1) + \log(x + 3) = 2$ **12.** $\log(x - 1) + \log(2x) = 2$
13. $\log(x - 1) - \log(x - 3) = 2$ **14.** $\log(x + 1) - \log(2x) = 2$

Solve the following equations for x; also, find a decimal approximation to the answer, to the nearest tenth.

15. $14.2 = 2^x$ **16.** $100 = 20^x$ **17.** $25 = x^4$ **18.** $18 = x^{2.3}$
19. $34 = 17^x$ **20.** $12 = 2^{x+3}$ **21.** $(x + 2)^4 = 200$ **22.** $45 = 5^{1-x}$
23. $25 = 8^{1/x}$ **24.** $146 = (x + 3)^{0.6}$ **25.** $41^{2x-1} = 2^x$ **26.** $172^{1-x} = 6^x$
27. $57^{x/2} = 5^{x+1}$ **28.** $0.88^{x+2} = 1.6^x$ **29.** $5^{x+1} = 3^{x-1}$ **30.** $2^{-x} = 6^{x+2}$
31. $\log_2 x = 0.33$ **32.** $\log_5 10 = x$ **33.** $\log_x 10 = 5$ **34.** $\log(x - 2) = 3$
35. $\log_3 30 = 2x$ **36.** $\log_{x-1} 18 = 2$ **37.** $\log_{2x} 14 = 3$ **38.** $\log_{2x} 8 = 3$

Solve the following equations for x.

39. $\log x^2 = (\log x)^2$ **40.** $\log 2^x = \log 3^{2x-1}$ **41.** $\log(\log x) = 3$
42. $\log(\log x^2) = 2$ **43.** $\log(\log 3x) = \log 2$ **44.** $\log^2 x - \log x^2 = 3$
45. $\log_2 x + \log_3 x = 5$ **46.** $\log 2^{x+1} + \log 3^x = 2$ **47.** $1 = e^x - 2e^{-x}$
48. $3 = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ **49.** $e^{2x} - 3e^x = 4$ **50.** $10^{x-1} = e^{x+1}$
51. $4 \log (\log x + 4) = 3$ **52.** $\ln x = \dfrac{8}{\ln x - 2}$

Use the formula $A = Pe^{it}$ in problems 53 to 62.

53. \$850 is deposited in an account paying 7.25% interest, compounded continuously. Find the amount in the account after $2\frac{1}{4}$ years.

54. \$2,000 is deposited in an account paying 5% interest, compounded continuously. Find the amount in the account after $3\frac{1}{2}$ years.

55. Find the amount that is an account after 4 years if the account pays 8.5% compounded continuously and if the initial principal was \$2,500.

56. \$45,000 is invested at a rate of 6% simple interest, compounded continuously. What is the value of the investment after $4\frac{1}{2}$ years?

57. How much money should be invested at 10% interest, compounded continuously, so that the value of the investment will be \$5,000 after 6 years?

58. How much money should be invested at 6% interest, compounded continuously, so that the value of the investment will be \$2,000 after 4 years?

59. At what interest rate does money double in 12 years, if the interest is compounded continuously?

60. At what interest rate does money triple in 12 years, if the interest rate is compounded continuously?

61. How long does it take the value in an account that pays 5% interest compounded continuously to triple in value?

62. How long does it take the value in an account that pays 7% interest compounded continuously to double in value?

Use the growth/decay formula $q = q_0 e^{rt}$ and the following information about carbon 14 in problems 63–68. The amount of carbon 14 in a living organism is constant while the organism is alive. At the death of the organism, the carbon 14 is not replenished and begins to decay. Thus, the percentage of the original amount of carbon 14 that is still present gives an estimate of the time that has passed since the organism died. Radioactive carbon 14 diminishes by radioactive decay according to the equation $q = q_0 e^{-0.000124t}$.

63. Compute the amount of carbon 14 still remaining in a sample that originally contained 100 milligrams (mg), after 8,000 years.

64. Compute the amount of carbon 14 still remaining in a sample that originally contained 10 milligrams (mg), after 3,500 years.

65. Estimate the age of a piece of charcoal (i.e., wood) if 30% of the original amount of carbon 14 is still present. That is, find t for which $q = 0.3q_0$.

66. Estimate the age of a piece of a sample in which 70% of the original amount of carbon 14 is still present.

67. What is the half-life of carbon 14?

68. After 1,200 years 18 μg (micrograms) of carbon 14 remained in a certain sample. How much was originally present?

Use the formula $\alpha = 10 \log \dfrac{I}{I_0}$ for problems 69–72.

69. The power of a sound increases by a factor of 20; that is, $I = 20I_0$. What is the resulting change in the decibel level?

70. The power of a sound decreases by a factor of 6; that is, $I = \frac{1}{6}I_0$. What is the resulting change in the decibel level?

71. How much must the power of a sound change to undergo a 3-decibel increase in intensity level? That is, solve for I if $\alpha = 3$.

72. How much must the power of a sound change to undergo a 5-decibel decrease in intensity level? That is, solve for I if $\alpha = -5$.

Use the formula $q = 1 - e^{-t}$ in problems 73–76.

73. What is the charge q on a capacitor after 3 RC time constants, to the nearest percentage? That is, find q when $t = 3$.

74. What is the charge q on a capacitor after 1.5 RC time constants, to the nearest percentage?

75. How many RC time constants are necessary for the charge on a capacitor to reach 45% of its full charge? That is, compute t for $q = 0.45$.

76. How many RC time constants are necessary for the charge on a capacitor to reach 90% of its full charge? That is, compute t for $q = 0.90$.

77. Solve $q = 1 - e^{-t}$ for t.

78. Allometry is the study of relationships between size and shape of organs of living animals, and the size and shape of organisms themselves. The equation of simple allometry is $y = \alpha x^{\beta}$, where α (alpha) and β (beta) are constants, and x and y describe size. Solve this equation for β.

79. Show that $b^x = e^{x \ln b}$ for $b > 0$.

80. The idea "n-factorial" is defined as $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$; for example, "3-factorial" is $3! = 1 \cdot 2 \cdot 3 = 6$, and $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$. An approximation formula, known as Stirling's formula, for factorials when n is large is $n! \approx \left(\dfrac{n}{e}\right)^n \sqrt{2\pi n}$. Compute the approximation value for $n = 30$, to the nearest unit.

81. A formula that arises in studying normal distributions in probability theory is $y = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$. Solve this equation for x.

82. In computer science, an AVL tree is a method of storing information in an efficient manner. It is a theorem of computer science that the height h of an AVL tree with N internal nodes is approximated by $h = 1.5 \log_2(N + 2)$. Solve this equation for N.

83. In testing a computer program in which the probability of a failure on any one test is $\dfrac{1}{h}$, the probability M that there will be no failure in N tests is $M = \left(1 - \dfrac{1}{h}\right)^N$. Solve this for N.

84. Refer to problem 83, with $h = 1{,}000$.
   a. What is the probability that there will be no failure in 2,000 tests?
   b. If we want the probability of a failure to be at least 0.01, how many tests would we expect to run? (That is, find N so that $M \geq 0.01$.)

85. Exponential functions are said to ''grow larger'' much faster than polynomial functions. Demonstrate this by computing the values for $f(x) = x^2$, a polynomial function, and for $g(x) = 2^x$, an exponential function for $x = 5$, 10, 20, and 40.

86. A method for computing values of e^x is given by the following equation: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$, where 2!, 3!, etc. are defined as shown in problem 80. The expression on the right (also called a power series) goes on forever, but one only needs a few terms for good accuracy for small values of x. Use the first six terms (five are given above) of this equation to compute an approximation for $e^{0.2}$. Check your answer with your calculator's $\boxed{e^x}$ key.

87. A problem from Mesopotamia, thousands of years old, asks how long it takes for money to double at 20% annually. The answer given is $3 + \dfrac{47}{60} + \dfrac{13}{60^2} + \dfrac{20}{60^3}$ (the Mesopotamians used a sexagesimal [base 60] system of numeration). How accurate is this answer? (Calculate how long it takes money to double at 20% annually yourself, then compare your answer to the Mesopotamian answer.)

88. It is estimated that 23% of a certain radioactive substance decays in 30 hours. What is the half-life of this substance?

89. The population of a bacteria was determined to increase by 15% in 15 hours. How long will it take the population to double, assuming it is increasing continuously at a fixed rate?

90. Assume that inflation has acted at a fixed rate for a 10-year period, and that in this time what initially cost one dollar now costs \$1.56. What was the rate of inflation for this period? How much did an item which initially cost one dollar cost after five years?

91. A population of bacteria is assumed to be growing continuously at a fixed rate. There are initially 10 μg (micrograms) of the bacteria; 2 hours later there are 40 μg.
   a. How many micrograms of bacteria will there be after 3 hours?
   b. When will there be 100 μg of bacteria?

92. The Richter scale was invented in 1935 by Charles F. Richter to measure the intensity of earthquakes. Each number on the scale represents an earthquake 10 times stronger than one of the next lower magnitude. For example, an earthquake of 5 on the Richter scale is 10 times stronger than one of measure 4 on the Richter scale. Suppose one earthquake measures 4.5 on the Richter scale, and a second measures 6.2. To the nearest unit how many times stronger is the second earthquake than the first?

93. In approximately the year 50 B.C., the Roman statesman Cicero was a governor in Asia Minor. He decided a case in which creditors had loaned money to the town of Salamis in Cyprus at 48% interest. Roman law permitted only 12% interest. Cicero decreed that only 12% compounded interest could be charged. On this basis, the deputies of Salamis determined that they owed 106 talents. Use the formula $A = P\left(1 + \dfrac{i}{n}\right)^n$ (from the previous section) to determine the original amount of the loan, using an interest rate of 12% compounded yearly, and assuming the loan was for 5 years.

94. Calculators often use the following method for computing $\log x$. Certain roots of 10 are stored:

$$10^{1/2} = 3.162278, \quad 10^{1/4} = 1.778279$$
$$10^{1/8} = 1.333521, \quad 10^{1/16} = 1.154782,$$
$$10^{1/32} = 1.074608, \quad 10^{1/64} = 1.036633, \text{ etc.}$$

Then, to compute, say, log 4, we find, by successive divisions, that

$$4 = 10^{1/2} \cdot 10^{1/16} \cdot 10^{1/32} \cdot 10^{1/128} \cdot 10^{1/2048}, \text{ etc., so}$$
$$\log 4 \approx 1/2 + 1/16 + 1/32 + 1/128 + 1/2{,}048 \approx 0.6021$$
(to four decimal places).

This method is reasonably efficient once enough roots of 10 are calculated and permanently stored into the calculator's memory. (To assure four-decimal place accuracy, 15 roots will suffice; on a typical eight-place calculator 32 roots will suffice.) Describe two ways in which this method could be modified to calculate natural logarithms.

Skill and review

1. Graph $f(x) = 2^{1-x}$.
2. Graph $f(x) = \dfrac{2}{(x-1)(x-5)}$.
3. Solve $|3 - 2x| < 13$.
4. Solve $\left|\dfrac{3-2x}{x}\right| < 13$.
5. Graph $f(x) = x^5 - 4x^4 + 2x^3 + 4x^2 - 3x$.
6. Simplify $\dfrac{2}{x-3} + \dfrac{2}{x+3} - \dfrac{5}{x+1}$.
7. Compute $(\frac{2}{5} - \frac{3}{4}) \div 2$.

Chapter 9 summary

- An exponential function is a function of the form $f(x) = b^x$, $b > 0$ and $b \neq 1$, with domain: $\{x \mid x \in R\}$ and range: $\{y \mid y \in R, y > 0\}$.
- **Logarithm** A logarithm is an exponent.
- **Equivalence of logarithmic and exponential form** $y = \log_b x$ if and only if $b^y = x$.
- **Logarithmic function** A function of the form $f(x) = \log_b x$, $b > 0$ and $b \neq 1$, with
  Domain: $\{x \mid x \in R, x > 0\}$
  Range: $\{y \mid y \in R\}$
- **Properties of exponential and logarithmic functions** (Assume $b > 0$ and $b \neq 1$):
  If $b^x = b^y$, then $x = y$
  If $\log_b x = \log_b y$, then $x = y$
  $y = \log_b x$ if and only if $b^y = x$
  $\log_b(xy) = \log_b x + \log_b y$
  $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
  $\log_b x^r = r \log_b x$
  $\log_b b = 1$
  $\log_b 1 = 0$
  $\log_b(b^x) = x$
  $b^{(\log_b x)} = x$
  $\log_b x = \dfrac{\log_m x}{\log_m b}$
  $\log_b x = \dfrac{\log x}{\log b}$
- **Common logarithms** Logarithms to the base 10 or $\log_{10} x$, which is abbreviated as simply $\log x$.
- **The symbol *e*** An irrational number; $e \approx 2.718$.
- **Natural logarithms** Logarithms to the base e or $\log_e x$, which is abbreviated as simply $\ln x$.

Chapter 9 review

[9–1]

1. Describe the behavior of the general exponential function with respect to the words "increasing" and "decreasing" and the value of b.

Use the properties of exponents to perform the indicated operations.

2. $5^{\sqrt{8}} \cdot 5^{\sqrt{50}}$
3. $\dfrac{4^{\sqrt{50}}}{4^{\sqrt{8}}}$
4. $(3^{\sqrt{2\pi}})^{\sqrt{8}}$

Graph the given function. State whether the function is increasing or decreasing. Label any intercepts.

5. $f(x) = 8^x$
6. $f(x) = 0.4^x$
7. $f(x) = 0.6^{-x}$
8. $f(x) = 4^{x+2}$
9. $f(x) = 3^{-x+1}$
10. $f(x) = 3^{x/2} - 1$

Solve the following exponential equations.

11. $9^x = 27$
12. $9^x = \sqrt{27}$
13. $10^{-2x} = 1{,}000$
14. $16^x = \frac{1}{8}$
15. $3^{x/2} = 9$
16. $(\sqrt{2})^x = 8$
17. $2^{-x} = 0.125$

[9–2] Find the value of the expression.

18. $\log_4 128$
19. $\log_{10} 0.001$
20. $3 \log_{25} \frac{1}{125}$
21. $5(3 \log_4 \frac{1}{8} + 2 \log_{10} 100)$
22. The following definition is incorrect; fix the logarithmic portion of the definition so it is correct. Definition:

$$\log_b x = y \text{ if and only if } x^y = b,\ b > 0,\ b \neq 1$$

Put each logarithmic equation into exponential form.

23. $\log_4 0.25 = -1$
24. $\log_5(x - 3) = 2$
25. $\log_2 y = 8$
26. $\log_3 9 = x + 2$
27. $\log_m x = y + 1$

Put each exponential equation into logarithmic form.

28. $x^3 = m - 3$
29. $4 = y^{2x} - 1$
30. $4 = y^{2x-1}$
31. $(x - 1)^y = 5$
32. $(x + 3)^{x+y} = y - 2$
33. $(5x)^2 = 3y$

Solve the following equations for x.

34. $\log_4 8 = x$
35. $\log_x 16 = 4$
36. $\log_{10}(x + 1) = -2$
37. $\log_x \frac{1}{8} = 3$
38. $\log_x k = 3$

Estimate the values of the following logarithms by stating two consecutive integers that bracket the value.

39. $\log_4 100$
40. $\log_{10} 15{,}600$

Simplify the following expressions.

41. $\log_5 5^m$
42. $x^{\log_x 5}$

[9–3] Solve the following logarithmic equations.

43. $\log_2 3x = -3$
44. $\log_5(x^2 - x) = \log_5 6$
45. $\log_3 \dfrac{5}{x} = \log_3(2x - 3)$
46. $\log_{0.5}(5x - 1) = -4$
47. $\log_2(-3x) = \log_2 \frac{1}{8}$
48. $\log_5(x - \frac{1}{2}) = \log_{10} 100$
49. $\log_3(x - 2) = \log_2 \frac{1}{16}$

Solve the following logarithmic equations.

50. $\log_5(x + 3) - \log_5(x - 1) = 2$
51. $\log_2 5 + \log_2(3 - 2x) = \log_2 x$
52. $\log_5 x - \log_5 3 = \log_5 2$
53. $\log_2(x - 4) + \log_2(x + 3) = \log_2(x^2 - 3x + 2)$
54. $\log_2 x^2 = 12,\ x > 0$
55. $\log_4 x^3 = \log_4 16^9,\ x > 0$

Rewrite the following expressions in terms of $\log_a x$, $\log_a y$, and $\log_a z$, for the given value of a.

56. $\log_2 8x^4y^2z$
57. $\log_4 \dfrac{8yz^3}{x^2}$
58. $\log_{10} \dfrac{x^5y^3z}{100}$

Assume $\log_a 2 = 0.3562$, $\log_a 3 = 0.5646$, and $\log_a 5 = 0.8271$. Use these values to find approximate values for the following logarithms.

59. $\log_a 30$
60. $\log_a 36$
61. $\log_a 0.2$

[9–4] Use a calculator to find the values of the following common logarithms. Round the results to 4 decimal places.

62. log 0.935
63. log 6,250
64. log 5,021,400,000,000,000,000,000
65. log 0.000 000 000 000 000 004 13

Compute the given logarithm. Round the results to 4 decimal places.

66. $\log_5 40$
67. $\log_{20} 1{,}000$
68. $\log_8 1.5$
69. $\log_{0.25} 20$

Use a calculator to find the values of the following natural logarithms. Round the results to 4 decimal places.

70. ln 12.31
71. ln 0.0035

Calculate the following values to 4 decimal places.

72. 10^π
73. $10^{-2.5}$
74. $e^{4.8}$

Graph the following logarithmic functions.

75. $f(x) = \log_5(x + 3)$
76. $f(x) = \log_2 2x$
77. $f(x) = \log_4 x - 3$
78. $f(x) = \log_2(-x)$

Simplify the following expressions.

79. $\log 10{,}000$
80. $\ln(e^{3x})^2$
81. $\ln e^{\sqrt{3}}$
82. $\log 10^{1-3x}$
83. $10^{\log 30}$
84. $e^{\ln 3x^2}$
85. $e^{\ln 2x} + \ln e^{2x}$
86. $\ln \sqrt{5}e^x$

87. \$2,500 is deposited in a bank account that computes its interest continuously. The simple interest rate is $6\frac{1}{4}\%$ per year. Use the formula $A = A_0e^{it}$ to compute the amount in the account after $3\frac{1}{2}$ years, where A is the amount after t years, A_0 is the initial deposit, and i is the simple yearly interest rate.

[9–5] Solve for x.

88. $8^x = 16^{4+5x}$
89. $(\sqrt{8})^{x-2} = 8^{3x}$

Solve the following equations for x, to the nearest hundredth.

90. $200 = x^{2.3}$
91. $80 = 2^{x+3}$
92. $45 = 9^{1-2x}$
93. $4.8 = 1.6^x$
94. $\log_2 x = 0.6$
95. $\log_5 30 = x$
96. $\log_x 100 = 5$
97. $\log_3 784 = 2x$
98. $\log_{2x-3} 3{,}000 = 6$

Solve the following equations for x.

99. $\log x^2 = \log^2 x$
100. $\log 2^{x-1} = \log 3^{2x-1}$
101. $\log(\log x) = 2$
102. $\log(\log x^2) = 1$
103. $\log(\log 3x) = \log 2$
104. $\log^2 x - \log x^2 = 15$
105. $\log_2 x + \log_3 x = 8$
106. $\log 3^{x+1} + \log 2^x = 10$
107. $3 = e^x - 4e^{-x}$
108. $5 = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
109. $5^{x-1} = e^{x+1}$
110. $4 \log x(\log x + 4) = 0$
111. $\ln x = \dfrac{12}{\ln x - 4}$

112. How much money should be invested at 10% interest, compounded continuously, so that the value of the investment will be \$5,000 after 6 years. Use the formula $A = Pe^{it}$.

113. At what interest rate does money double in 9 years, if the interest is compounded continuously?

114. It is estimated that 23% of a certain radioactive substance decays in 30 hours. What is the half-life of this substance?

Chapter 9 test

Use the properties of exponents to perform the indicated operations.

1. $\dfrac{8^{\sqrt{18}}}{8^{\sqrt{32}}}$
2. $(2^{\sqrt{2}})^{\sqrt{8}}$

Graph the given function. State whether the function is increasing or decreasing. Label any intercepts.

3. $f(x) = 3^x$
4. $f(x) = 0.3^x$
5. $f(x) = 3^{x+2}$
6. $f(x) = 3^x - 1$

Solve the following exponential equations.

7. $9^x = 3$
8. $27^x = \sqrt{27}$
9. $25^x = \frac{1}{125}$
10. $(\sqrt{5})^x = 25$
11. $100^{-x} = 0.001$

Find the value of the expression.

12. $\log_8 128$
13. $3 \log_2 \frac{1}{16}$
14. $2(3 \log_8 \frac{1}{8} + \log_{100} 10)$

15. The following definition is incorrect; fix the *exponential* portion of the definition so it is correct.

$\log_a x = y$ if and only if $x^y = a$, $a > 0$, $a \neq 1$

Put each logarithmic equation into exponential form.

16. $\log_5 0.25 = -x$
17. $\log_3(x - 3) = 2$
18. $2 \log_m x = y$

Put each exponential equation into logarithmic form.

19. $2^5 = 32$
20. $(x - 1)^3 = m$
21. $z = y^{2x-1}$

Solve the following equations for x.

22. $\log_4 \frac{1}{8} = x$
23. $\log_{10}(2x - 5) = 3$
24. $\log_{x+1} \frac{1}{8} = 3$

Graph the following logarithmic functions.

25. $f(x) = \log_3 x$
26. $f(x) = \log_5(x + 1)$
27. $f(x) = \log_2 x - 1$

Solve the following logarithmic equations.

28. $\log_2 \dfrac{x}{2} = -3$
29. $\log_2(x^2 - 14x) = \log_2 32$
30. $\log_3(x + 3) = \log_3(4x - 12)$
31. $\log_3(5x - 1) = -2$
32. $\log_3(3x - 2) = \log_4 \frac{1}{16}$
33. $\log_2(x + 6) - \log_2(x - 1) = 5$
34. $\log_5(x + 1) - \log_5 3 = \log_5 2$
35. $2 \log_{10}(x + 2) = \log_{10}(x + 14)$
36. $\log_2 x^3 = 15$, $x > 0$

Rewrite the following expressions in terms of $\log_a x$, $\log_a y$, and $\log_a z$, for the given value of a.

37. $\log_3 \dfrac{9x^3y}{z^4}$
38. $\log_{10} \dfrac{x^{10}y^3z}{1,000}$

Assume $\log_a 2 = 0.3562$, $\log_a 3 = 0.5646$, and $\log_a 5 = 0.8271$. Use these values to find approximate values for the following logarithms.

39. $\log_a 12$
40. $\log_a 40$
41. $\log_a 0.4$

Use a calculator to find the values of the following logarithms. Round the results to 4 decimal places.

42. $\log 31,020,000,000,000$
43. $\log 0.000\,000\,001\,03$
44. $\log_5 50$
45. $\ln 1,000$

Calculate the following values to 4 decimal places.

46. $10^{\pi-2}$
47. $3^{\sqrt{e}}$

Simplify the following expressions.

48. $\ln[(e^x)^2]$ **49.** $(\ln e^x)^2$ **50.** $\log 10^{18}$

51. $10^{\log 27}$ **52.** $e^{\ln 5x} + \ln e^{5x}$

53. \$2,500 is deposited in a bank account that computes its interest continuously. The simple interest rate is $7\frac{1}{2}\%$ per year. Use the formula $A = A_0e^{it}$ to compute the amount in the account after $3\frac{1}{4}$ years.

Solve the following equations for x, to the nearest hundredth.

54. $27^x = 9^{4+5x}$ **55.** $178 = x^{1.9}$ **56.** $80 = 3^{x-2}$

57. $\log_5 x = 3$ **58.** $\log_2 12 = x$ **59.** $\log_x 34 = 2.5$

Solve the following equations for x.

60. $\log^3 x = \log x^4$ **61.** $4^{x-2} = 3^{6x-1}$

62. $\log_3(\log_2 x) = 2$ **63.** $\log(\log x^2) = 1$

64. $\log^2 x - \log x^2 = 35$ **65.** $\log_2 x + \log_3 x = 3$

66. $9 = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ **67.** $3^{x-1} = e^{x+1}$

68. How much money should be invested at 6% interest, compounded continuously, so that the value of the investment will be \$2,000 after 4 years. Use the formula $A = Pe^{it}$.

69. At what interest rate does money double in 12 years, if the interest is compounded continuously? Round to the nearest 0.1%. (See the formula from problem 68.)

70. The half-life of a certain radioactive substance is 30 hours. How much of an initial amount of 25 grams of this substance will remain after 15 hours? Use $q = q_0e^{rt}$, and round the result to the nearest 0.1 grams.

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